

Research Note

A note on gravitational wave lensing

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Abstract. In a recent paper (Ruffa 1999) it was proposed that the massive black hole at the Galactic center may act as a gravitational lens focusing gravitational wave energy to the Earth. Considering the gravitational wave signal emitted by galactic spinning pulsars, an enhancement in the gravitational wave intensity by a factor of a few thousand is found. For galactic and extra-galactic sources the intensity enhancement can be as high as 4000 and 17 000, respectively. In this note we consider the probability of significant signal enhancement from galactic and extra-galactic pulsars by the proposed mechanism and find that it is actually negligible. The lensing effect due to a possible companion object (a star or the galactic center black hole) of the gravitational wave source is also investigated in the framework of the classical microlensing theory.

Key words. gravitation – gravitational waves – gravitational lensing

1. Introduction

Gravitational lensing of electromagnetic waves is a well known phenomenon predicted by the General Theory of Relativity (for a review see Schneider et al. 1992). In principle, gravitational lensing of gravitational waves should occur in the same way as it does for light. The most obvious difference is that gravitational wave propagation is not disturbed by dust grains, as happens for light, so that the central part of our galaxy may be investigated by using the next generation of gravitational wave detectors.

In a very interesting paper Ruffa (1999), assuming that the mass of the Galactic center is in the form of a massive black hole with mass $M \simeq 2.6 \times 10^6 M_{\odot}$ (for a Galactic center overview see Eckart et al. 2002), the gravitational wave lensing problem was studied by a typical Fraunhofer diffraction approach. It was pointed out (Ruffa 1999) that extra galactic sources can be amplified by a factor of about 17 000 and galactic neutron stars by over 4000. The author also argued that the Earth would take about 10.1 days to traverse the focused region of the extra galactic sources. For galactic bulge sources the focused region would be scaled down by a factor of nearly 3 and hence the observing time would be reduced to about 3 days.

The question that naturally arises is “how likely is it that we see the proposed enhancement?” To answer this question we need to estimate the number of sources that could be

expected to be observed, both galactic and extra-galactic. In principle, of course, there are many extra-galactic sources as there are $\sim 10^{11}$ galaxies. However, we need to limit the number, applying a cut-off by requiring that the expected intensity (after amplification) be greater than the sensitivity of the detector.

As will be seen, the chances of seeing the dramatic enhancement calculated by Ruffa are extremely small. This is due to the fact that Ruffa assumes a very special geometry, with the source, lens and observation point aligned. Whereas Ruffa considered Fraunhofer diffraction, an analysis using “geometric optics” for gravitational waves had been undertaken, in which the special geometry of Ruffa was not assumed (De Paolis et al. 2001). This analysis gave a much higher probability but a lower enhancement.

In this paper, we evaluate the probability of enhancement of the gravitational wave signals as a consequence of diffraction by massive compact objects in the highly aligned geometry.

We also consider the lensing effect due to a possible companion star of the gravitational wave source in the framework of the classical microlensing theory. In addition to this case we consider the gravitational wave lensing due to the massive black hole at the galactic center. As we shall see, the gravitational wave signal amplification due to the companion object (or the black hole at the galactic center) might be detectable by the VIRGO detector for some orbital parameters of the binary system.

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2. Probability of signal enhancement by the galactic central black hole

Let us assume that the observer, the lens and the source (a spinning pulsar) are on the same line. The lens will focus the gravitational waves that pass through the Einstein radius

$R_E = \sqrt{\frac{4GM}{c^2} \frac{Dd}{D+d}}$ to the Earth in a region orthogonal to the line of sight. Here D is the distance between the observer and the lens (≈ 8 kpc) and d is the lens-source distance. Using diffraction theory, the focused area is determined by D and the diffraction pattern for the circular annular region bounded by radii R_E and $R_E + \delta R_E$. Note that a gravitational wave that passes at the distance $R_E + \delta R_E$ from the lens is focused in $D + \delta D$. It is possible to show (by evaluating the 3 dB points of the signal, see Ruffa 1999) that the radius of the focused region is given by

$$\rho(d) \simeq 0.175\lambda \frac{D}{R_E(d)} \quad (1)$$

where λ is the gravitational wave length¹. As pointed out by Ruffa, the increase in intensity depends on the ratio (known as the intensity ratio) of the annular region bounded by the radii R_E and $R_E + \delta R_E$ to the focused region area. This can be determined by the system of equations²

$$\begin{cases} \rho(d)(D + \delta D) \simeq R_E(d)\delta D, \\ \delta R_E(d) = \frac{1}{R_E(d)} \frac{2GM}{c^2} \frac{d^2}{(D+d)^2} \delta D. \end{cases} \quad (2)$$

We note that the first equation is Ruffa's Eq. (4) at first order approximation (since $\rho \ll R_E$) and the second one derives from differentiating the expression for the Einstein radius R_E with respect to D . Solving for δD , we finally obtain

$$\delta R_E(D) \simeq \frac{1}{2} \frac{\rho d}{D+d}, \quad (3)$$

which reduces to $\delta R_E(D) \simeq \rho/2$ for $d \gg D$. Taking into account the previous equations and the expression of the Einstein radius, the intensity ratio turns out to be

$$IR(d) \simeq \frac{d}{D+d} \frac{R_E}{\rho}, \quad (4)$$

implying a gravitational wave amplification factor³

$$A = \sqrt{IR(d)}. \quad (5)$$

Assume now that Ruffa's treatment still holds if the source is not fully aligned with the observer-lens line but forms at least an angle $\theta \simeq \rho(d)/d$ with it. Consequently, the probability density of finding the source within a circular region of radius between ρ and $\rho + d\rho$ is proportional to the ratio between that

¹ In the case of non-precessing pulsars rotating with frequency of ~ 30 Hz, the emitted gravitational waves has frequency ~ 60 Hz (see e.g. Shapiro & Teukolsky 1983). Consequently, the gravitational wave length turns out to be $\sim 5 \times 10^8$ cm.

² We note that in all case of interest $R_E \sim 10^{17}$ cm and, from Eq. (1), $\rho \ll R_E$.

³ Since the gravitational wave amplitude h is proportional to the square root of the energy flux, the enhancement in h is proportional to the square root of the intensity ratio.

circular area and the surface of the cylinder centered at the galactic center and having radius d and height H . From simple geometric considerations, this probability density, after integration over $d\rho$, gives the following probability of finding a source within the radius $\rho(d)$

$$P_g(d) \simeq 3 \times 10^{-3} \frac{c^2}{GM} \lambda^2 \frac{D(D+d)}{d^2 H}. \quad (6)$$

On the other hand, the real probability to observe a gravitational wave enhancement in the case of the observer, lens and source on the same line of sight has to take into account also the number of expected sources behind the central black hole.

From an extrapolation of the neutron star (NS) birthrate (Narayan & Ostriker 1990)⁴ and from the number of supernovae required to account for the heavy element abundance in the Milky Way (Arnett et al. 1989), we expect that our galaxy contain $N \simeq 10^9$ NSs. For simplicity we assume that the NS population has a usual disk-like shape, i.e. the number density profile is given by

$$n(r, z) \simeq \frac{\Sigma_{NS}}{2Hm} e^{-z/H} e^{-(r-D)/\Delta}, \quad (7)$$

where $\Sigma_{NS} \simeq 1 M_\odot \text{ pc}^{-2}$, $m \simeq 1 M_\odot$, $H \simeq 0.30$ kpc and $\Delta \simeq 3.5$ kpc is the typical length scale of the galactic disk. In order to consider the distribution of matter behind the black hole, Eq. (6) has to be multiplied by the factor $n(r, 0)V/N$, V being the total volume of the disk. Consequently, the probability to have an enhancement of the gravitational wave signal according to the mechanism proposed by Ruffa is

$$P(d) \simeq 10^{-2} \frac{\Sigma_{NS}}{Nm} \frac{c^2}{GM} \frac{\lambda^2 \Delta^2}{H} \frac{D(D+d)}{d^2} e^{-(d-D)/\Delta} \text{ yr}^{-1}. \quad (8)$$

In Fig. 1 we give the probability P as a function of the source distance d from the galactic center. As one can see, the probability to have a gravitational wave enhancement *a la Ruffa* by the black hole at the galactic center from a pulsar in the line of sight to the galactic center is as low as $\sim 10^{-20}$ per year. The total probability (assuming that there are no pulsars closer than 5×10^{-5} kpc to the galactic center) is 10^{-12} per year. In other words, we could expect to wait for a hundred times the age of the universe to see one!

Let us see now what happens in the case of a gravitational wave signal due to a supernova explosion in other galaxies behind the galactic black hole. Since $d \gg D$, in this case $\rho \simeq 0.175\lambda \sqrt{Dc^2/4GM}$. The probability of a galaxy being within the focused region at a distance d is simply $(\rho/2d)^2$. Thus the total probability will go as $(\rho/d_{\min} - \rho/d_{\max})/4$. Since

⁴ From historical supernovae (SNe) in our galaxy one gets an SN frequency of one event every 40 ± 10 yr (Tammann et al. 1994). These estimates do not take into account any SN rate evolution with time and the fact that only about 10% of the historical SNe have been close and bright enough to have been detectable by the naked eye, and therefore have to be considered as a lower limit to the inferred present galactic SN number. A more realistic estimate of the SN rate for our galaxy is about one over 20 years (Panagia 1999). Observations show that large late-type spiral galaxies produce about 10 SNe per century (Tammann et al. 1994).

Table 1. The maximum flux amplification $A_{f,\max}$, the gravitational wave signal amplification $A_{h,\max}$ and the microlensing event time-scale $T_{1/2}$ are shown for different values of the orbital radius R_{orb} , period P and inclination angle i , and the companion mass object M . Here we assume $m = 1.4 M_{\odot}$. In the last three lines the black hole at the galactic center is assumed to be the lens object around which a gravitational wave emitting pulsar rotates.

$M (M_{\odot})$	$R_{\text{orb}} (\text{cm})$	$P(h)$	i°	$A_{f,\max}$	$A_{h,\max}$	$T_{1/2} (h)$
1.4	2×10^{11}	8	0.1	1.46	1.21	4.4×10^{-3}
1.4	2×10^{11}	8	0.001	116.73	10.80	1.1×10^{-4}
1.4	2×10^{13}	8×10^3	0.1	1.0004	1.0002	3
1.4	2×10^{13}	8×10^3	0.001	11.7	3.4	8×10^{-2}
10	2×10^{13}	4×10^3	0.1	1.013	1.006	1.55
10	2×10^{13}	4×10^3	0.001	31.20	5.58	4×10^{-2}
10	2×10^{14}	1.2×10^5	0.1	1.00018	1.00009	45
10	2×10^{14}	1.2×10^5	0.001	9.9	3.14	1.31
2.6×10^6	3×10^{16}	5.1×10^5	1	1.03	1.02	2047
2.6×10^6	3×10^{16}	5.1×10^5	0.5	1.21	1.10	1210
2.6×10^6	3×10^{16}	5.1×10^5	0.1	4.14	2.04	378

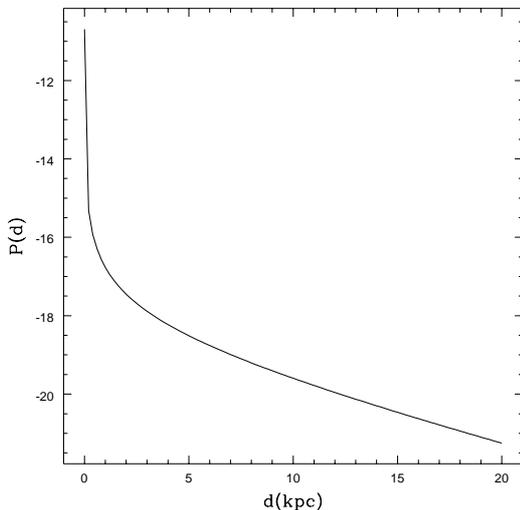


Fig. 1. The probability of observing the gravitational wave signal amplitude enhancement by the black hole at the galactic center from a galactic disk pulsar is shown as a function of the source distance from the lens.

$d_{\max} > d_{\min}$, we only need to find the probability of a galaxy at the closest distance to be on the line of sight, i.e. the number density of galaxy does not matter. It is known that supernovae explode in typical galaxies less frequently than once every 25 years. Consequently, the relevant probability is $\sim 0.01\rho/d_{\min}$ per year. Taking d_{\min} as 1 Mpc, the probability is less than $\sim 10^{-15}$ per year! As such, there is no real likelihood of being able to see the Ruffa enhancement. Of course, there are also other expected sources of bursts of gravitational waves from gamma-ray bursts (see e.g. Ruffert & Janka 1998) or binary systems (see e.g. De Paolis et al. 2002 and references therein) that could in principle be detectable by VIRGO (Jaranowski & Królak 1999; Passaquieti 2000), LIGO (Barish 2000) or LISA (Hiscock et al. 2000; Vecchio 1999) experiments. However, from the discussion above, it can easily be calculated that the expected probability of seeing a gravitational wave amplification by the central black hole of one of such event is even less than $\sim 10^{-15}$ per year.

3. Signal enhancement in binary systems

Let us consider now the enhancement of the gravitational waves emitted by a rotating neutron star due to the microlensing effect of a companion star moving around the compact source on a circular orbit with radius R_{orb} . Let m and M be the masses of the pulsar and of the companion star, respectively. We further assume that the orbital plane of the binary star is inclined at an angle i with respect to the line of sight. The companion star coordinates, with respect to a reference frame with x and y axes on the orbital plane, are given by

$$x_M = R_{\text{orb}} \cos \theta, \quad y_M = R_{\text{orb}} \sin \theta, \quad z_M = 0, \quad (9)$$

where $\theta = \sqrt{G(m+M)/R_{\text{orb}}^3} t$ is the orbital angle measured with respect to the positive direction of the x axis pointing towards the observer. The distance between the companion star M and the line of sight to the gravitational wave source turns out to be

$$d_{\text{ls}}(\theta) = \sqrt{x_M^2 \sin^2 i + y_M^2}. \quad (10)$$

For a source-observer distance greater than the distance between the source and the lens ($\sim R_{\text{orb}}$), the Einstein radius is $R_E \simeq \sqrt{4GM R_{\text{orb}}/c^2}$. Consequently, as in usual microlensing theory, the flux amplification factor is given by (see e.g. Schneider et al. 1992)

$$A_f = \frac{u^2 + 2}{u \sqrt{u^2 + 4}}, \quad (11)$$

with $u = d_{\text{ls}}(\theta)/R_E$. Obviously, since the impact parameter u depends on the position of the companion star on the orbit, we expect that the flux amplification A_f is a periodic function of time with period $P = 2\pi \sqrt{\frac{R_{\text{orb}}^3}{G(m+M)}}$. We also note that the effect of the amplification is maximal for $i \rightarrow 0^\circ$. The expected gravitational wave amplitude enhancement is $A_h \simeq \sqrt{A_f}$. In Table 1, assuming $m = 1.4 M_{\odot}$, the maximum flux amplification $A_{f,\max}$, the gravitational wave signal amplification $A_{h,\max}$ and the microlensing event time-scale $T_{1/2}$ are shown for different values

of the orbital parameters (R_{orb} , P , i and M). As one can note, the amplification effect decreases for increasing inclination angles while the event time-scale $T_{1/2}$ goes up. Since the gravitational wave amplification is the same for a compact or a normal companion star, assuming that VIRGO will be able to detect only 1% of the pulsars with spin period $P_{\text{sp}} < 100$ ms within about 25 kpc from Earth in three years of integration, one expects $\sim 10^4$ detectable binaries. Therefore, the number of binary systems with plane inclination angle $i \leq 0.1^\circ$ is easily found to be about 10. Of course, since the probability of having high enough amplification of the signal for long enough is expected to be rather small, the chances of having a really detectable amplified signal from a binary system are low. Some higher probability is expected if the lens object is the black hole at the galactic center, due to its higher mass. In this case, as one can see from Table 1, both the signal amplification and duration are expected to be high enough to have a signal detectable by VIRGO. For details about the detectable gravitational wave amplitude see Gourgoulhon & Bonazzola (1996).

Finally, we want to note that the effects of the gravitational light deflection and amplification in eclipsing binary stars has been considered in detail (Marsh 2001; Beskin & Tuntsov 2002) and by Schneider (1989) for the binary pulsar 1957+20 as a way for determining the binary system parameters. Actually, the gravitational wave amplification is, if gravitational wave astronomy becomes practicable in the future, a much more powerful tool since gravitational waves are never shielded by the lens object (as happens for light rays) giving the unique possibility to trace the sources even in regions (like the galactic center) where light suffers strong absorption.

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