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Lunar numerical theory and determination of parameters k_2, δ_M from analysis of LLR data

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Abstract. Analysis of lunar ranging data of 1970–2001 was carried out to estimate values of the lunar phase angle $\delta_M = 0.0446 \pm 0.0001$ and Love's number $k_2 = 0.0205 \pm 0.0001$. The Moon's dynamic theory has been constructed by simultaneous numerical integration of equations of motion of the major and minor planets and the Moon with equations of lunar librations. Partial derivatives with respect to estimated parameters were obtained following the method of arbitrary constant varying.

Key words. astrometry – celestial mechanics – ephemerides – Moon

1. Introduction

The most precise lunar observations nowadays are the lunar laser ranging (LLR) measurements. A large amount of these very accurate data is the basis for construction of the modern lunar ephemeris. Analysis of LLR data makes it possible to determine and improve a set of selenodynamical parameters. Among them are the Moon's dynamical flattenings $\beta = (C-A)/B$ and $\gamma = (B-A)/C$. LLR data combined with lunar artificial observations permit to obtain the value for C/MR^2 and draw conclusions about the mass distribution and the existence of a lunar liquid iron core.

The goal of the paper is to improve the lunar tidal parameters δ_M and k_2 and the dynamical model from analysis of a 30 year set of LLR observations using the new ephemeris EM-3.

2. Analysis of the observations

2.1. Lunar ephemeris EM-3

The most precise and prospective method for a construction of the lunar ephemeris is a numerical integration of equations of orbital and rotational lunar motion. In the framework of the lunar and planetary theory EPM-2000 developed at the Institute of Applied Astronomy of the Russian Academy of Sciences (Pitjeva 2001) the preliminary theory of lunar orbital-rotation motion EM-1 has been constructed (Aleshkina et al. 1997). It fits modern lunar laser ranging (LLR) observations with an accuracy of 2 ns for post-fit residuals, which corresponds to 30 cm in distance from the station to the reflector.

The lunar rotation is computed from differential equations of second order for the Euler angles ϕ, θ, ψ:

$$\ddot{\phi} = \frac{\dot{\omega}_x \sin\psi + \dot{\omega}_y \cos\psi + \dot{\theta}\left(\dot{\psi} - \dot{\phi}\cos\theta\right)}{\sin\theta}$$

$$\ddot{\theta} = \dot{\omega}_x \cos\psi - \dot{\omega}_y \sin\psi + \dot{\phi}\dot{\psi}\sin\theta \qquad (1)$$

$$\ddot{\psi} = \dot{\omega}_z - \ddot{\phi}\cos\theta + \dot{\theta}\dot{\phi}\sin\theta$$

ϕ is the angle along the Earth's equator J2000 from the equinox J2000 to the line of the nodes with the lunar equator; θ is the inclination of the lunar equator to the Earth's equator J2000; ψ is the sidereal angle along the lunar equator. These angles specify a frame rotating with the Moon in an inertial coordinate system. The equations are transformed from the well-known systems of kinetical and dynamical Euler's equations with respect to the angular velocity of the Moon $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$ and its derivatives.

In the rotating frame the Euler's dynamical equations for the angular momentum are as follows:

$$\frac{\mathrm{d}(I\omega)}{\mathrm{d}t} = \boldsymbol{T} - \omega \times I\omega \qquad (2)$$

where $\boldsymbol{T}$ is a torque due to perturbing bodies and I is the lunar inertia tensor consisting of the principal moments A, B, C.

In connection with an extended time span of observations and the necessity of an improvement of the dynamical model used, the amelioration of the lunar ephemeris was continued. The lunar ephemeris EM-2 was modeled (Aleshkina et al. 2000) taking into account the following set of additional perturbations:

1. Tidal interaction in the Earth-Moon system and its effect on a lunar rotation, causing variations with time of I and gravitational harmonics.

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2. Addition of 300 minor planets by means of a simultaneous numerical integration.

Taking into account the tidal interaction, the torque $\boldsymbol{T}$ can be presented as a sum of the torque $\boldsymbol{M}$ (M_x, M_y, M_z) without tides and the tidal torque $\boldsymbol{N}$ (N_x, N_y, N_z); Eq. (2) assume the form:

$$\begin{aligned}\dot{\omega}_x &= \frac{\gamma-\beta}{1-\beta\gamma}\omega_z\omega_y + \frac{M_x}{A} + \frac{N_x}{C}\\ \dot{\omega}_y &= \beta\omega_z\omega_x + \frac{M_y}{B} + \frac{N_y}{C}\\ \dot{\omega}_z &= -\gamma\omega_x\omega_y + \frac{M_z}{C} + \frac{N_z}{C}.\end{aligned} \quad (3)$$

If R is the lunar mean radius of 1737.4 km, m and $\boldsymbol{r}$ the mass and radius-vector of the perturbing body in the rotating system, $r_{\rm p}$ the radius-vector of the testing point and P_2 is the second-degree Legendre polynomial, then tidal perturbations in lunar rotation are as follows:

- changes in lunar moments of inertia and potential owing to tides from the Earth and the Sun; they are connected with the additional tide-raising potential $U_{\rm t}$ (Getino & Ferrandiz 1991):

$$U_{\rm t} = mk_2\frac{R^5}{r_{\rm p}^5}Gm\frac{r_{\rm p}^2}{r^3}P_2\left(\cos\left(\boldsymbol{r}_{\rm p}, \boldsymbol{r}\right)\right); \quad (4)$$

- changes in lunar moments of inertia and potential owing to the Moon's rotation, which are connected to the rotational potential U_r:

$$U_{\rm r} = -\frac{1}{2}mk_2\frac{R^5}{r_{\rm p}^5}(\boldsymbol{\omega}, \boldsymbol{r}_{\rm p})^2. \quad (5)$$

Calculations of the tidal torques were carried out according to expressions derived in (Getino & Ferrandiz 1991; Newhall & Williams 1997; Krasinsky 1999), when the lunar inertia tensor consists of the diagonal rigid-body component I_0 and two time-varying components accounting for tidal deformation of the lunar figure $I_{\rm t}$ and deformation due to the rotation of the Moon $I_{\rm r}$.

Tidal torque $\boldsymbol{N}$ can be represented as:

$$\boldsymbol{N} = \boldsymbol{N}_{\rm t} + \boldsymbol{N}_{\rm r} + (I_{\rm t} + I_{\rm r})\boldsymbol{\omega}\times\omega - \frac{\rm d}{{\rm d}t}(I_{\rm t} + I_{\rm r})\boldsymbol{\omega} \quad (6)$$

where $\boldsymbol{N}_{\rm t} = \boldsymbol{r}\times{\rm grad}\,U_{\rm t}, \boldsymbol{N}_{\rm r} = \boldsymbol{r}\times{\rm grad}\,U_{\rm r}$.
All terms on the right side of the Eq. (6) are proportional to the lunar Love's number k_2 and its sensitivity to the analysis of LLR data comes from this part of the differential equations. In connection to the existence of the tidal time delay for a nonelastic body the time-varying parts of the inertia tensor $I_{\rm t}, I_{\rm r}$ should be calculated at an earlier moment than t, namely $(t-\tau)$. We used the phase angle $\delta_M = \omega\tau$ instead of τ and one part of (6) depends on its value.

The dynamical flattenings $\beta = (C-A)/B, \gamma = (B-A)/C$ are related to C_{20}, C_{22} as:

$$\begin{aligned}\beta &= \frac{(4C_{22} - 2C_{20})}{(4C_{22} + 2C_{20}) + 2C/MR^2}\\ \gamma &= 4C_{22}\frac{MR^2}{C}\end{aligned} \quad (7)$$

Table 1. Maximum differences of the lunar distance, longitude and librations between LE403 and EM-3.

	Unit	LE403 - EM-3
r	m	0.600
λ	arcs	0.007
ϕ	arcs	0.500
θ	arcs	0.200
ψ	arcs	0.500

where M and R are the mass and mean radius of the Moon respectively.

In the equations of motion the coefficients C_{nm}, S_{nm} are divided by C/MR^2 only. We have fixed the value of C/MR^2 = 0.390689526131941 and fitted the parameters C_{nm}, S_{nm}.

For integration the values of $\beta = 6.316769\times10^{-4}, \gamma = 2.280043\times10^{-4}$ were taken from the IERS Standards (1992) and the corresponding values of C_{20}, C_{22} were computed according to the above expressions. Hence, the independent parameters in the integration are β, γ, C/MR^2; C_{20}, C_{22} are derived.

The tidal bulge on the Earth due to the Moon appears with a delay having the phase angle $\delta_{\rm E}$. In numerical integration the influence of the bulge on the lunar acceleration $\ddot{\boldsymbol{r}}$ is taken into account with:

$$\ddot{\boldsymbol{r}} = -\frac{3k_{2\rm E}GM}{r^3}\left(1+\frac{M}{M_{\rm E}}\right)\left(\frac{R_{\rm E}}{r}\right)^5\begin{pmatrix} x + y\delta_{\rm E}\\ y - x\delta_{\rm E}\\ z\end{pmatrix} \quad (8)$$

$M_{\rm E}, R_{\rm E}$ are the mass and radius of the Earth, $\boldsymbol{r}(x, y, z)$ the geocentric radius-vector of the Moon and $k_{2\rm E}$ the Earth's Love number. The corresponding value for the tidal lunar deceleration $\dot{n}_M$ is calculated in accordance with the well-known relation:

$$\dot{n}_M = -4.5\left(\frac{M}{M+M_{\rm E}}\right)\left(\frac{R_{\rm E}}{r}\right)^5 n_M^2 k_{2\rm E}sin\,(2\delta_{\rm E}) \quad (9)$$

$n_M = 2.6617\times10^{-6}$ rad/s is the lunar mean motion.

Comparisons of the ephemeris EM-1 and EM-2 show that taking into account tidal effects significantly improves the theory (Aleshkina et al. 2000). Differences in lunar geocentric distances, computed from the two theories, are 50–80 cm for a 15 year time span. The discrepancy between amplitudes in Euler's angles of physical libration ranges up to 2″–4″.

After an increase of 15 percent in the number of observations, the extended version of the lunar ephemeris EM-3 for 1970-2001 has been worked out using the dynamical model from EM-2.

Figure 1 shows the discrepancies between EM-1 without tidal effects and EM-3 including tidal effects.

The numerical ephemeris EM-3 fits modern observations for 1985–2000 with post-fit residuals of 0.3–0.4 ns and is in good agreement with the JPL lunar ephemeris LE403.

The discrepancies between LE403 (Standish et al. 1995) and EM-3 are presented in Table 1.

The integration was carried out by Everchart's method with the automatical change of integration step

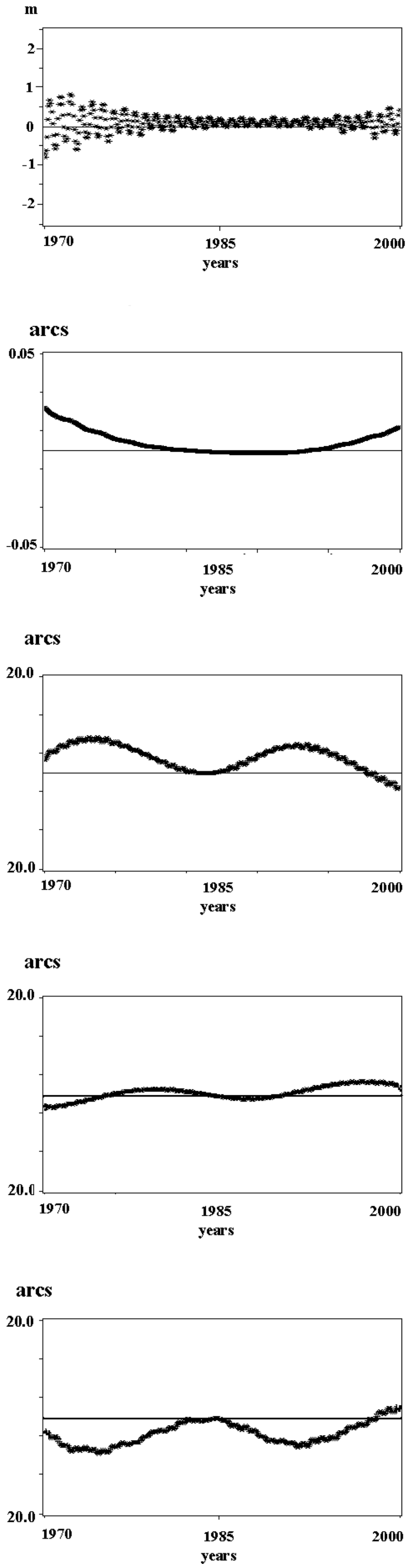

Fig. 1. EM-1–EM-3, comparison for lunar **a)** distances, **b)** longitudes and **c)**–**e)** librations angles ϕ, θ, ψ respectively.

(Krasinsky & Vasilyev 1997). The starting epoch for the integration was JD 2 446 000.5 (October 10, 1984). To compute partial derivatives with respect to parameters, the variational equations were simultaneously integrated. Partial derivatives with respect to estimated parameters δ_M and k_2 were computed following the method of arbitrary constants. Dynamical models and observational information are presented in Table 2.

2.2. Observations used

LLR observations 1970.03.16–2002.02.05, made at 5 laser stations of the three observatories (McDonald, Haleakala, CERGA) were analyzed to improve the parameters of the dynamical model used and the values of δ_M and k_2. The number of observations is 14 638, 9 were rejected according with the 3 σ criterion. The number of estimated parameters of the model is 49. These are the following:

1. coordinates and velocities of the Moon for the initial date 2 446 000.5 (corresponding to 1984.10.27);
2. initial values for lunar librations and its time derivatives for 2 446 000.5;
3. coordinates of the LLR-stations and reflectors (excluding the longitude and latitude of Apollo-15);
4. delay angle δ_{E} of the tidal bulge of the Earth;
5. Stokes' coefficients of the selenopotential for harmonics up to 4th order.

The corrections to the LLR-station coordinates were computed according to the IERS Standards 1996 (McCarthy 1996) with the plate motions in ITRF96 (Boucher et al. 1998).

During the whole period of LLR observations two different ranging systems based on two laser's modifications and timing systems were used (Dickey et al. 1994) and their accuracies are presented in Table 3, Col. 4. Besides the main solution with 49 estimated parameters mentioned above a more complete analysis was carried out with the following additional parameters: time shift, secular trends in latitude for the stations with respect to MLRS2, linear trends in sidereal time and lunar sidereal time, secular trends in obliquity and precession. Mean post-fit residuals for the main and additional solutions obtained with EM-3 are presented in Table 3 in Cols. 6, 7 respectively. For CERGA observations 1997.01.01–1998.06.01 it is necessary to add +0.7 ns as was pointed out in Chapront-Touzé et al. (1999). The post-fit residuals for CERGA observations are shown in Fig. 2. Comparison of the two solutions shows almost a factor of two difference in rms with the additional set of parameters. Formal errors of estimated parameters k_2, δ_M in the main solution are 1.5 times greater than in the additional one.

3. Determination of parameters

3.1. Parameters of the model EM-3

After 4 iterations fitting the observations, the following values of estimated parameters of the dynamical model were obtained.

Table 2. Lunar ephemeris EM.

Theory	Time span	Number of obs.	Post-fit errors (cm)	Dynamical model
EM-1 (1997)	1970–1996	10 000	30	Integration of Sun, 9 planets, Moon's motion and rotation without tidal effects
EM-2 (2000)	1970–1998	11 924	6	Integration of Sun, 9 planets, 300 asteroids, Moon's motion and rotation with tidal effects
EM-3 (2002)	1970–2001	14 638	8	Integration of Sun, 9 planets, 300 asteroids, Moon's motion and rotation with tidal effects

Table 3. Fit of observations.

Station	Time span	Number of obs.	Apriori errors (cm)	Pre-fit residuals (cm)	Post-fit1 residuals main, (cm)	Post-fit2 residuals add., (cm)
MacDonald	1970–1985	3451	25	40	40	30
MLRS1	1986–1988	275	15	6	8	4
MLRS2	1988–2002	2187	3	8	8	7
CERGA (ruby las.)	1984–1986	1160	15	15	12	10
CERGA (neod.las.)	1987–2002	6871	3	8	8	4
Haleakala	1985–1990	694	3	8	8	6

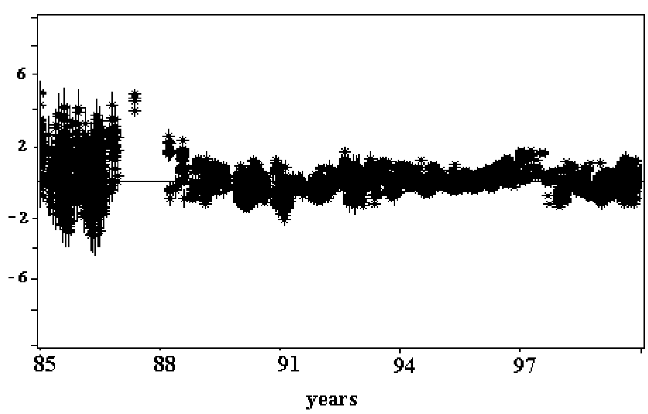

Fig. 2. Post-fit residuals for CERGA observations, global solution.

Reflector coordinates are presented in Table 4. The longitude and latitude of Apollo-15 are fixed.

$\delta_E = (9004.33'' \pm 2.61'') = 0.04365 \pm 0.00001$

Stokes' coefficients of the lunar gravitational field:

$C_{20} = (2.02351 \pm 0.000013) \times 10^{-4}$
$C_{21} = (0.0175 \pm 0.0007) \times 10^{-7}$
$C_{22} = (2.2699 \pm 0.00006) \times 10^{-5}$
$S_{22} = (0.0017 \pm 0.0001) \times 10^{-7}$
$C_{30} = (-8.0680 \pm 0.0008) \times 10^{-6}$
$C_{31} = (3.7067 \pm 0.0011) \times 10^{-5}$
$S_{31} = (4.8400 \pm 0.0010) \times 10^{-6}$
$C_{32} = (3.3963 \pm 0.0001) \times 10^{-6}$
$S_{32} = (1.1861 \pm 0.0012) \times 10^{-6}$
$C_{33} = (1.0962 \pm 0.0002) \times 10^{-6}$
$S_{33} = (-1.8727 \pm 0.0001) \times 10^{-7}$.

Table 4. Selenocentric coordinates of reflectors.

Reflector	X (km)	Y (km)	Z (km)
Apollo-11	1591.9550	690.7244	21.0049
Apollo-14	1652.6968	−520.9723	−109.7286
Apollo-15	1554.6751	98.1201	765.0067
Lunochod-2	1339.3489	801.8934	756.3595

3.2. Lunar parameters δ_M and k_2

The goal of the present paper is the estimation of the lunar parameters δ_M and k_2 from the analysis of residuals. The adopted values for these parameters used in integration of the motion equations are as follows:

$k_2 = 0.01919$

$\delta_M = 0.0429 = 8856.6''$.

Corrections to these starting values were obtained using the least square method.

$\Delta k_2 = 0.00135 \pm 0.00008$
$\Delta\delta_M = 0.0017 \pm 0.0001 = (347.8 \pm 17.2)''$.

The value of $k_2 = 0.0205$ is close to $k_2 = 0.0215$ obtained by Nakamura (1983) and differs from $k_2 = 0.0302$ (Dickey et al. 1994), which was used in the construction of the LE403 ephemeris.

Dissipation in the Moon can be characterized by the factor Q that is related to phase angle δ in following way:

$Q = 1/\text{tg}(2\delta)$.

With $\delta_M = 0.0446$ obtained in our analysis, Q_M is as follows:

$Q_M = 11.18$.

This estimate Q_M corresponds to the lunar solid friction for a one month period and is close to the Earth Q. The theoretical value of Q for a rigid planet (Zharkov & Trubitsyn 1978) is of the order of 100. The low Earth's value is due to turbulent dissipation in shallow seas. For the Moon such a low estimation is not explained.

4. Conclusions

It should be mentioned that the effects of the lunar core were not included in the dynamical model, although it can affect the results, but not significantly.

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