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The oblique pulsator model revisited

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Abstract. The oblique pulsator model accounts for most of the pulsation properties of the rapidly oscillating Ap (roAp) stars. The model predicts that modes are seen as equidistant multiplets separated by the angular frequency of rotation. The relative amplitudes of the components may be calculated and directly compared with observations. The effects of rotation introduce amplitude asymmetry, that is peaks corresponding to azimuthal numbers m and $-m$ are unequal. In this paper we propose improvements to the model that consist of including effects of the centrifugal force and in using a non-perturbative treatment of the magnetic field influence. We show that in roAp stars the centrifugal force is the primary source of the rotational frequency shift. Although the amplitude asymmetry arises from the Coriolis force, its size is strongly affected by the centrifugal force. For dipole modes ($\ell = 1$) we develop a simple geometrical picture of pulsation in the presence of rotation and a magnetic field. We provide some numerical results for a representative model of roAp stars which is applied to the case of HR 3831. We find that the mode that agrees with the observed amplitude ratios in this star significantly departs from alignment with the magnetic axis. We discuss problems posed by the observational data of HR 3831, emphasizing difficulties of the standard oblique pulsator model which assumes that the excited mode is nearly aligned with the magnetic field.

Key words. stars: oscillations – stars: magnetic fields – stars: rotation

1. Introduction

The rapidly oscillating Ap (roAp) stars are high order p-mode pulsators. The pulsation periods are in the 5–15 min range which is nearly the same as that of the solar oscillations. However, the mode amplitudes in roAp stars, with typical values of one millimagnitude, are higher by three orders of magnitude than those in the Sun. Kurtz (1982) argued that the magnetic field must play an essential role in roAp oscillations as the maxima of the oscillation amplitudes coincide with the maxima of the longitudinal field. The properties of the magnetic field in roAp stars are similar to that in the whole group of Ap stars. The observed field is predominantly dipolar and has a kiloGauss strength. Since their discovery two decades ago (Kurtz 1978), the number of roAp stars has grown to 32. Most of the pulsation data in roAp stars may be interpreted in terms of rotating dipole modes that Kurtz assumed to be symmetric around the magnetic axis. The model provides a natural explanation of the observed multiplets in the spectrum of the oscillations in which the components are split by exactly the frequency of rotation. Dziembowski & Goode (1985) generalized the oblique pulsator model bytaking into account effects of the Coriolis force. The signature of this effect is an inequality in the amplitudes of the side peaks, which is in fact observed. In this model, the amplitude differences depend on the ratio of rotational to magnetic frequency perturbation. Hence the amplitudes yield a constraint on the internal magnetic field. The generalized model was further developed by Kurtz & Shibahashi (1986) who gave analytical relations between amplitudes of the multiplets in the case of a dominating magnetic field over rotational effects. Some additional improvements have been brought by Shibahashi & Takata (1993) and Takata & Shibahashi (1995). In all these works, effects of the centrifugal force were ignored and those of the magnetic field were treated as a small perturbation. Neither of these approximations are justified in the case of roAp stars. Our aim here is to eliminate these shortcomings.

In Sect. 2, we consider the dynamical effects of the magnetic field and rotation on arbitrary oscillation modes. We treat effects of rotation as a perturbation of magneto-acoustic modes. Since rotation couples nearly degenerate states of different m values, the degenerate perturbation formalism is used. It leads to a matrix eigenvalue problem for mode frequencies and relative amplitudes of spherical harmonics of different m's. In Sect. 3, we consider individual modes in the observer's system. Each mode is seen as a multipletwith $(2\ell+1)$ components.

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We determine their relative amplitudes. In Sect. 4, we explore the consequences of these improvements for the mode properties in the special case of $\ell = 1$. Finally in Sect. 5, we apply these results to a representative model of roAp stars. We also discuss problems posed by the roAp star HR 3831 for which the inequality of the side peaks is clearly seen.

2. Rotational perturbation of magneto-acoustic oscillations

In roAp stars the observed surface magnetic fields have kilo-Gauss intensities. Even at these large values, the field has almost no dynamical effect on acoustic wave propagation except in a thin layer at the surface of the star (one or two percent in radius), where the magnetic pressure is comparable to (or larger than) the gas pressure. Therefore, in this thin layer the magnetic field effect cannot be treated as a perturbation. Non-perturbative treatments of the effects of a magnetic field on oscillations in roAp stars were developed by Dziembowski & Goode (1996), Bigot et al. (2000) and Cunha & Gough (2000). It is known from these previous studies that the magnetic field distorts the modes so that the angular dependence is in general described by a linear combination of spherical harmonics of different ℓ's and the same m (as long as the field is axisymmetric). For a pure dipole field the combination involves only ℓ's of the same parity. We adopt here the approach developed by Bigot et al. (2000) with one additional simplification. This simplification consists of assuming that the angular dependence for individual modes may be approximated by a single spherical harmonic in the magnetic reference system. The effects of rotation on p-mode oscillations may be regarded as small perturbations. For the rotation periods of roAp stars, which are of the order of days or more, the angular velocity of rotation, Ω, satisfies the strong inequality

$$\Omega \ll \omega_{\rm dyn} = \sqrt{\frac{GM}{R^3}}. \tag{1}$$

If the rotation is the only non-spherical perturbation, then in the coordinate system with the polar axis aligned with the rotation axis, each of the individual modes is described by a single spherical harmonic. In that case the frequency shift due to rotation may be written in the following form (e.g. Gough & Thompson 1990; Dziembowski & Goode 1992)

$$\delta\omega = mC_{n\ell}\Omega + Q_{\ell m}D_{n\ell}\Omega^2, \tag{2}$$

where

$$\begin{aligned} Q_{\ell m} &= 2\pi \int_{-1}^{1} \left(Y_\ell^m\right)^* (\cos\theta) P_2(\cos\theta) Y_\ell^m(\cos\theta) {\rm d}(\cos\theta) \\ &= \frac{\Lambda_\ell - 3m^2}{4\Lambda_\ell - 3} \quad \text{with} \quad \Lambda_\ell = \ell(\ell+1). \end{aligned} \tag{3}$$

The first term in Eq. (2) is due to the Coriolis force. The coefficient $C_{n\ell}$ is called the Ledoux constant (Ledoux 1951). The coefficient $D_{n\ell}$ represents the radial integral of the effect of the centrifugal distortion of the star on the mode (see Appendix). We made two approximations regarding the quadratic term. First, we ignore spherical change of the star due to the surface average of the centrifugal force. It causes a small m-independent frequency shift, which is not of interest to us. The second approximation is to include only the effect of the centrifugal distortion, which is much larger than the second-order effect of the Coriolis force. The reason why in roAp stars, which are rather slow rotators, the quadratic term of rotation may exceed the linear one is a consequence of high radial order n of p-modes excited in these stars. For such modes we have approximately $D_{n\ell} \sim n$ and $C_{n\ell} \sim 1/n^2$. The ratio of the second to the first term in Eq. (2) is $\sim n^3\Omega/\omega_{\rm dyn}$ and is greater than 1 for all roAp stars for which we have data to evaluate it.

The previous approach to calculate the effects of rotation is not directly applicable to roAp stars. The reason is that in most of these stars the magnetic and rotation axes are tilted by a certain angle β, called the obliquity. The combined magnetic and rotational perturbations do not have axial symmetry. It will be convenient for us to consider this joint perturbation in the reference system with the polar axis aligned with the magnetic axis, which we will call the magnetic system.

We allow the rotational frequency perturbation to be of the same order as the frequency separations between magnetic eigenmodes of different m's. The perturbation due to rotation is non-axisymmetric which implies coupling of modes of different m's. We are thus in the situation requiring the use of a degenerate perturbation theory. Individual eigenmodes are no longer described by a single value of m. Simultaneous effects of rotation and inclined magnetic field have already been investigated (e.g. Dicke 1982; Dziembowski & Goode 1985; Gough & Thompson 1990). In this paper, we will follow the approach of Dziembowski & Goode but we improve it in two different respects. The magnetic field effects are treated with a non-perturbative approach. We also take into account the effect of the centrifugal distortion of the star for the reasons already mentioned.

2.1. A degenerate perturbation theory

In this subsection, we use the reference system with the polar axis aligned with the magnetic axis. The zeroth order equation is

$$(\omega_{|m|}^{\rm mag})^2 \boldsymbol{\xi}_m = \mathcal{L}(\boldsymbol{\xi}_m) + \mathcal{B}(\boldsymbol{\xi}_m) \equiv \mathcal{M}(\boldsymbol{\xi}_m), \tag{4}$$

where $\boldsymbol{\xi}_m$ represents the displacement, $\omega_{|m|}^{\rm mag}$ the magnetic eigenfrequency, $\mathcal{L}$ the usual adiabatic oscillation operator and $\mathcal{B}$ the Lorentz force operator (see e.g. Unno et al. 1989). The $\mathcal{B}$ operator introduces a dependence on $|m|$. Indeed, the degeneracy with respect to m which exists in absence of symmetry breaking agents is partially removed by the magnetic field since it breaks the spherical symmetry of the star. To simplify notation, we do not put the ℓ and n subscripts. Numerical solutions of Eq. (4) for more-or-less realistic models of roAp stars were obtained by Bigot et al. (2000) and by Cunha & Gough (2000) for polytropic models. The eigenfrequencies are complex. The non-Hermitian nature of the operator is a consequence of the boundary condition applied at the base of the magnetic layer which implicitly assumes an efficient dissipation of the downward propagating Alfvénic waves. This complex nature of the problem will be ignored in our considerations here. In fact, the only quantities obtained from Eq. (4) that we will use here are differences between eigenfrequencies $\omega_{|m|}^{\rm mag}$ at specified ℓ and n. As we have already pointed out, we assume a single spherical harmonic dependence for individual magneto-sonic modes.

Now we consider effects of rotation. The eigenvalue equation becomes

$$\omega^2 \boldsymbol{\xi} = \mathcal{M}(\boldsymbol{\xi}) + \mathcal{R}(\boldsymbol{\xi}), \tag{5}$$

where $\mathcal{R}$ denotes the perturbing rotational operator.

In agreement with the properties of Ap stars we assume that the rotation and magnetic axes are tilted, hence the perturbation due to rotation is non-axisymmetric. This leads to coupling of the magnetic states corresponding to different azimuthal orders m. The strength of the coupling depends on the size of the rotational shift of the frequency relative to the frequency differences, $|\omega_m^{\rm mag} - \omega_{m'\neq m}^{\rm mag}|$. Even with typical magnetic fields found in roAp stars, such as 1 kG, the rotational shift of frequency, dominated by the centrifugal distortion, can be comparable to the above frequency differences. Therefore, a standard perturbation theory cannot be applied in this model. We have to use a degenerate perturbation theory as Dziembowski & Goode (1985) did in the same context. We then consider the following displacement vector

$$\boldsymbol{\xi} = \sum_{m=-\ell}^{\ell} \alpha_m \boldsymbol{\xi}_m, \tag{6}$$

which takes into account the $(2\ell + 1)$ coupled magnetic levels. The $\boldsymbol{\xi}_m$ are solutions of Eq. (4) and α_m are coefficients to be determined. The sum involves only nearly degenerate modes, which in the adopted approximation are described by a single value of ℓ. The orthogonality of $\boldsymbol{\xi}_m$ follows from orthogonality of Y_ℓ^m. We assume the completeness of the set of $\boldsymbol{\xi}_m$.

Using Eq. (6) in Eq. (5), after multiplication by $\boldsymbol{\xi}_j^*$ and integration over the volume, we get the eigensystem,

$$\sum_{m=-\ell}^{\ell} \alpha_m \left\{ O_{jm} - \omega^2 \delta_{jm} \right\} = 0 \qquad j = -\ell, ..., \ell \tag{7}$$

with

$$O_{jm} = \left(\omega_{|m|}^{\rm mag}\right)^2 \delta_{jm} + R_{jm} \quad R_{jm} = \int_{\mathcal{V}} \boldsymbol{\xi}_j^* . \mathcal{R}(\boldsymbol{\xi}_m) \rho {\rm d}V. \tag{8}$$

The $(2\ell + 1) \times (2\ell + 1)$ symmetric matrix O is a sum of a diagonal matrix built with eigenvalues of Eq. (4) and a matrix whose elements are projections of the operator $\mathcal{R}$ onto the base $\boldsymbol{\xi}_m$.

Therefore, to calculate the effect of an oblique rotation on magnetic states, one has to find the $(2\ell+1)$ eigenfrequencies ω^2 and the corresponding eigenvectors α_m of the matrix O. The condition for non-trivial solutions requires that the determinant of Eq. (7) vanishes. This condition yields the eigenfrequencies.

2.2. Calculation of the R_{jm} elements

The elements of matrix $\mathcal{R}$ are given by

$$R_{jm} = Z_{jm}\Omega + W_{jm}\Omega^2, \tag{9}$$

where

$$Z_{jm} = 2\, i\, \omega_0 \int_{\mathcal{V}} \boldsymbol{\xi}_j^* . (\boldsymbol{e}_\Omega \times \boldsymbol{\xi}_m) \rho {\rm d}V, \tag{10}$$

with ω_0 the frequency in absence of magnetic field and rotation, $\boldsymbol{e}_\Omega$ a unit vector along rotation axis, and

$$W_{jm} = D_{n\ell} \int Y_\ell^j(\theta,\phi)^* P_2(\theta_R) Y_\ell^m(\theta,\phi) {\rm d}(\cos\theta){\rm d}\phi. \tag{11}$$

In the last expression θ_R is the polar angle in the rotation system.

The Z_{jm} coefficients couple components with $j = m, m \pm 1$ because $\boldsymbol{e}_\Omega$ expressed in magnetic system contains only terms proportional either to $\sin\phi$ or $\cos\phi$. Using the complex spherical harmonic property $Y_\ell^{j*} = (-1)^j Y_\ell^{-j}$, it is easy to show that

$$Z_{-j-m} = (-1)^{j+m+1} Z_{jm}. \tag{12}$$

The W_{jm} coefficients couple components with $j = m$, $m \pm 1$, $m \pm 2$ because $P_2(\theta_R)$ expressed in the magnetic system generates harmonics with $|m| \leq 2$. The property of the integral in Eq. (11) leads to the relation

$$W_{-j-m} = (-1)^{j+m} W_{jm}. \tag{13}$$

After integration the explicit expression for R_{jm} becomes

$$\begin{aligned} \frac{R_{jm}}{2\omega_0} = & \left(mC_{n\ell}\Omega P_1^0(\beta) + Q_{jm} D_{n\ell}\Omega^2 P_2^0(\beta) \right) \delta_{j,m} \\ & -H_m \left(C_{n\ell}\Omega P_1^1(\beta) - \frac{(2m+1)D_{n\ell}\Omega^2}{4\Lambda_\ell - 3} P_2^1(\beta) \right) \delta_{j,m+1} \\ & -H_{-m} \left(C_{n\ell}\Omega P_1^1(\beta) - \frac{(2m-1)D_{n\ell}\Omega^2}{4\Lambda_\ell - 3} P_2^1(\beta) \right) \delta_{j,m-1} \\ & -\frac{D_{n\ell}\Omega^2}{4\Lambda_\ell - 3} P_2^2(\beta) \left(H_m H_{m+1} \delta_{j,m+2} + H_{-m} H_{-m+1} \delta_{j,m-2} \right), \end{aligned} \tag{14}$$

where

$$H_m = \frac{\sqrt{(\ell - m)(\ell + m + 1)}}{2}. \tag{15}$$

The P_ℓ^m's are the usual associated Legendre functions. This matrix is non-diagonal as long as the rotation and magnetic axes are not aligned ($\beta \neq 0$).

In the calculation of the R_{jm} elements we made an additional approximation which consists in neglecting the contribution from the thin magnetic layer to the radial integrals in the $C_{n\ell}$ and $D_{n\ell}$ coefficients.

3. A single mode seen as a multiplet in the observer's system

In order to calculate fluctuations of luminosity in the observer's system, we now express the eigensolutions of Eq. (7) in the reference system with the polar axis directed towards the observer. Following Dziembowski & Goode (1985), we assume that the intensity fluctuation $\delta I/I$ has the same angular dependence as fluctuation of photospheric pressure $\delta p/p$. In the magnetic reference system the intensity fluctuations is written

$$\left(\frac{\delta I}{I}\right) \propto \sum_{m=-\ell}^{\ell} \alpha_m Y_\ell^m(\theta_B, \phi_B) \mathrm{e}^{\mathrm{i}\omega t}. \tag{16}$$

From the well-known relation for spherical harmonic transformation (e.g. Edmonds 1960)

$$Y_\ell^m(\theta_B, \phi_B) = \sum_{k=-\ell}^{\ell} d_{mk}^{(\ell)}(\beta) Y_\ell^k(\theta_R, \phi_R), \tag{17}$$

we write the intensity fluctuations in the rotation system

$$\left(\frac{\delta I}{I}\right) \propto \sum_{m=-\ell}^{\ell} a_m Y_\ell^m(\theta_R, \phi_R) \mathrm{e}^{\mathrm{i}\omega t}. \tag{18}$$

The coefficients a_m and α_m obey the relation

$$a_m = \sum_{j=-\ell}^{\ell} \alpha_j d_{jm}^{(\ell)}(\beta). \tag{19}$$

The expression for the luminosity fluctuations in observer's system is then obtained by two transformations. One consists in writing the fluctuations in an inertial reference system with the same polar axis as the rotation axis but which differs by the longitudes, i.e.

$$\left(\frac{\delta I}{I}\right) \propto \sum_{m=-\ell}^{\ell} a_m Y_\ell^m(\theta_I, \phi_I) \mathrm{e}^{\mathrm{i}(\omega - \mathrm{m}\Omega)t}, \tag{20}$$

where we adopted $\theta_R = \theta_I$ and $\phi_R = \phi_I - \Omega t$. The final transformation consists in a rotation of the coordinate system so that the new polar axis coincides with the line of sight

$$\left(\frac{\delta I}{I}\right) \propto \sum_{m'=-\ell}^{\ell} \sum_{m=-\ell}^{\ell} a_m d_{mm'}^{(\ell)}(i) Y_\ell^{m'}(\theta_O, \phi_O) \mathrm{e}^{\mathrm{i}(\omega - \mathrm{m}\Omega)t}. \tag{21}$$

The subscripts B, R, I, O refer to the magnetic, rotational, inertial and observer systems, respectively. In the last expression i denotes the angle between the rotation axis and the line of sight. Finally, each eigenmode is seen by an observer as a $(2\ell + 1)$-component multiplet. In the time domain, such a mode will be observed as an amplitude modulated pulsation with a modulation period equal to $2\pi/\Omega$.

Unfortunately, from observations we can get only an averaged luminosity over the visible disk. Then, only the components with $m' = 0$ survive in the disk-average intensity fluctuations. Thus, the relative luminosity change seen by the observer has the from of the multiplet

$$\left(\frac{\delta L}{L}\right) \propto \sum_{m=-\ell}^{\ell} A_{\ell,m} \cos{(\omega - m\Omega)t}, \tag{22}$$

where the amplitudes are given by

$$A_{\ell,m} \propto d_{m0}^{(\ell)}(i)\, a_m = d_{m0}^{(\ell)}(i) \sum_{j=-\ell}^{\ell} \alpha_j d_{jm}^{(\ell)}(\beta). \tag{23}$$

The mode is represented by a multiplet as long as its geometry is described by more than one value of m in the rotation system.

It is convenient to use the following amplitude ratios (Kurtz & Shibahashi 1986),

$$\gamma_m^- = \frac{A_{\ell,m} - A_{\ell,-m}}{A_{\ell,m} + A_{\ell,-m}} \qquad \gamma_m^+ = \frac{A_{\ell,m} + A_{\ell,-m}}{A_{\ell,0}} \tag{24}$$

which are the observables of interest. In order to evaluate these observables, we need a stellar model as well as information about rotation and magnetic field. We now have pretty good ideas about the model for specific stars. The rotation rate Ω is the most accurately determined parameter, but the inclination angle i is never reliably known. Regarding the magnetic field, we are in a worse situation. In all calculations of magnetic effects on oscillations of roAp stars a simple dipole model has adopted. In this case the field is fully characterized by its polar value at the surface, B_p, and the obliquity angle β. We have observational assessments of B_p but with uncertainties. The only justification for using a dipole model of the magnetic field is its simplicity. Therefore the main application of our observables is to subject the model to an observational test.

From Eq. (23) we get for the amplitude ratios,

$$\gamma_m^- = \frac{|a_m| - |a_{-m}|}{|a_m| + |a_{-m}|} \tag{25}$$

and

$$\gamma_m^+ = \left|\frac{d_{m0}^{(\ell)}(i)}{d_{00}^{(\ell)}(i)}\right| \frac{|a_m| + |a_{-m}|}{|a_0|}. \tag{26}$$

Note that γ_m^- measures the departure from equality of side peak amplitudes at $+m$ and $-m$. It is important to note here that the asymmetry of the amplitudes A_m in the observer's system comes from the asymmetry of the coefficients of the coupling, $|\alpha_m| \neq |\alpha_{-m}|$. Neither the magnetic field nor the centrifugal distortion can explain this inequality since they affect the components $+m$ and $-m$ in the same way. In our problem, only the Coriolis force affects in a different way these two components. Ignoring this force, the problem would have a mirror symmetry that is invariant to the transformation $m \rightarrow -m$. The formal proof follows from Eq. (13) which implies that in absence of the Coriolis force $\alpha_{-m} = (-1)^m \alpha_m$ and thereby $\gamma_m^- = 0$. The symmetry of $d_{jm}^{(\ell)}$ implies the same relation for the a_m-coefficients. The value of γ_m^+ depends both on the aspect angle i (first factor) and relative role of rotation and magnetic field (second factor).

4. Application to $\ell = 1$

The dipole mode is of special importance because it is dominant in roAp stars. Furthermore, these modes admit a simple geometrical interpretation.

4.1. The mode polarization

In the present case, equating to zero the determinant of Eq. (7) leads to the following cubic equation for eigenfrequencies

$$\sigma^3 - (1+\mu)\sigma^2 + (\mu s^2 - \chi^2)\sigma + \chi^2(1+\mu c^2) = 0, \tag{27}$$

where

$$\sigma = \frac{\omega - \omega_1^{\rm mag}}{D} + \frac{1}{3}, \tag{28}$$

and

$$\chi = C/D \qquad \text{with} \qquad D = 3/5 D_{n1}\Omega^2, \quad C = C_{n1}\Omega. \tag{29}$$

The obliquity angle appears in $s = \sin\beta$ and $c = \cos\beta$. The magnetic field effects come only through the parameter

$$\mu = \frac{\omega_0^{\rm mag} - \omega_1^{\rm mag}}{D}, \tag{30}$$

which, according to the results in Bigot et al. (2000), is always negative. We should stress that Cunha & Gough (2000) found situations where μ is positive. We are unable to explain this disagreement.

The geometrical picture of the $\ell = 1$ modes is a displacement of the sphere representing the stellar surface. In the absence of a magnetic field, the $m = 0$ modes represent displacements along the rotation axis, z. The $m = \pm 1$ modes represent motions of the sphere along a circle in the (x, y) plane with two opposite directions. To describe the modification of the geometrical picture due to the combined effect of magnetic field and rotation, we consider radial component of the displacement $\xi_{\rm r}$ at the star surface. Since we are in the framework of the linear pulsation theory the horizontal displacement is irrelevant. We oriented the axes so that the magnetic field axis lies in the (x, z)-plane. The radial displacement has the same angular and temporal dependence as in Eq. (18). In Cartesian coordinates (x, y, z), this is written

$$\xi_r \propto (f_3 z - f_1 x)\cos\omega t + f_2 y \sin\omega t, \tag{31}$$

where,

$$f_1 \equiv \frac{a_1 - a_{-1}}{\sqrt{2}} = -sc\mu\sigma,$$
$$f_2 \equiv \frac{a_1 + a_{-1}}{\sqrt{2}} = -sc\mu\chi,$$
$$f_3 \equiv a_0 = \sigma^2 - \chi^2 - \mu\sigma s^2. \tag{32}$$

The displacement vector described Eq. (31) lies in a plane whose normal makes an angle δ with the rotation axis and is given by

$$\delta = \arctan\left(\frac{f_3}{f_1}\right) = \arctan\left(\tan\beta - \frac{\sigma^2 - \chi^2}{sc\mu\sigma}\right). \tag{33}$$

The inclination of the plane is mainly determined by the relative size of the centrifugal and magnetic shifts. Indeed, in roAp stars only the centrifugal distortion effects can be comparable to the magnetic ones. The role of the Coriolis force is marginal regarding the inclination of the mode.

We now rotate the coordinate system around the y-axis by the angle δ. In the new coordinate system (X, y, Z) the displacement is writen

$$\xi_r \propto F_1 X \cos\omega t - f_2\, y \sin\omega t \tag{34}$$

with,

$$F_1 = f_1/\cos\delta. \tag{35}$$

During the pulsation cycle, the maximum of the displacement vector describes an ellipse in the (X, y) plane. The three solutions of Eq. (27) correspond to three different polarizations of the motion which are described by the parameter

$$\psi = \arctan\left(\frac{f_2}{F_1}\right). \tag{36}$$

Its value does not depend on the choice of the reference system. The special cases are:

$\psi = 0$ – linear polarization along the X-axis,
$\psi = \pm\pi/2$ – linear polarization along the y-axis,
$\psi = \pm\pi/4$ – circular polarization.

The geometry of the problem is illustrated in Fig. 1. The Coriolis force plays an essential role in the polarization of the displacement vector. Indeed, as we have already mentioned only this force creates unequal coefficients for the components $m = \pm 1$ and thereby $f_2 \neq 0$, which leads to an elliptical polarization. When it is neglected ($f_2 = 0$), we have three linearly polarized modes along the three orthogonal axes of the reference system except for the singularity at $F_1 = 0$, as we have for $\beta = 0$.

In absence of rotation two modes are degenerate with the same frequency, $\omega_1^{\rm mag}$. If we take into account only the effect of the centrifugal distortion as a manifestation of the rotation, it would raise this degeneracy as long as $\beta \neq 0$, but it would also lead to equal amplitudes for the coefficients $|a_{\pm 1}|$ since the centrifugal distortion does not introduce asymmetry. These two modes would be therefore linearly polarized along two orthogonal axes.

The effects of the Coriolis force and the mode ellipticity decrease if the rotation axis is approaching the mode plane ($|\delta| \to \pi/2$). When the rotation axis is inside the mode plane ($\delta = \pm\pi/2$), the effects of the Coriolis force vanish and the mode is linearly polarized.

The three dipole modes are completely characterized by the (σ, δ, ψ) parameters. In Fig. 2 these parameters are plotted as functions of the obliquity angle β for three different values of μ which depends on magnetic field strength. For better visualization, we have selected $\chi = 0.1$, though the realistic values for roAp stars are generally smaller ($\chi \sim 0.01$). The fourth quantity $\mathcal{D}$ will be introduced in Sect. 4.2. The efficiency of the coupling between m-components of the mode by rotation depends on the size of the frequency separation of the unperturbed magnetic modes compared with the rotational shift (Coriolis + centrifugal). Two regimes exist depending on the value of μ.

For small values of $|\mu| \ll 1$, the rotational (centrifugal) effects dominate over the magnetic effects. In the frame with

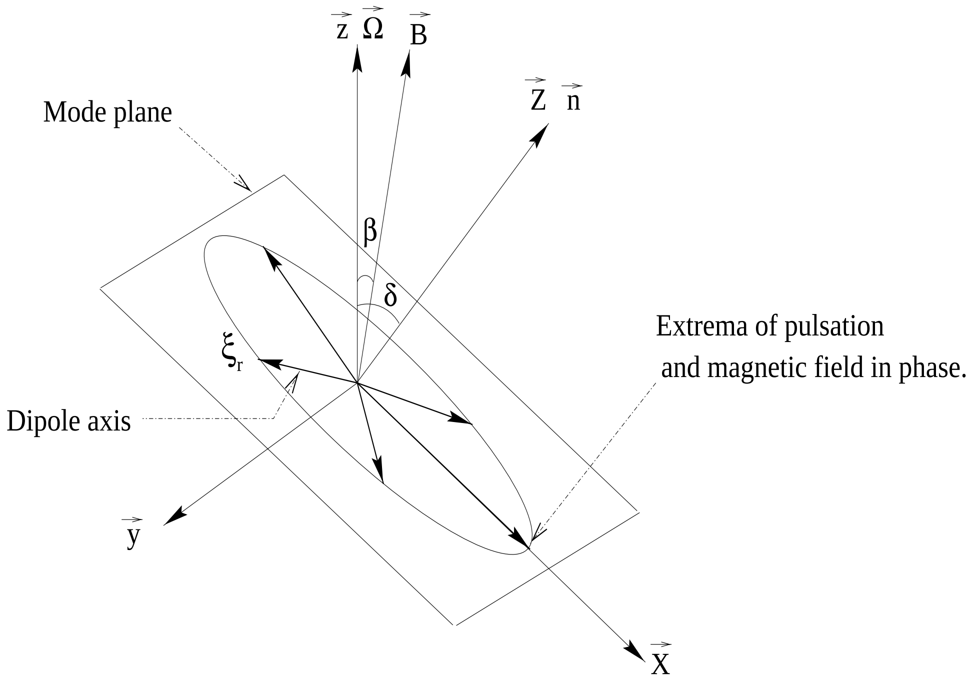

Fig. 1. Representation of the dipole mode geometry. During one periode of oscillation, the maximum of the displacement vector ξ_r describes an ellipse in the mode plane whose normal ***n*** makes an angle δ with the rotation axis. The line which joins the two angular extrema of ξ_r is the axis of symmetry of the dipole mode. The thick arrows correspond to different positions of the dipole axis during the pulsation cycle. The magnetic and rotation axes are in the same plane (X, Z) but they are tilted by the angle β. For this figure we plot $\delta = 40$ deg and $\beta = 10$ deg which are the values found for HR 3831, see Sect. 5. For clarity, we draw on this plot an ellipticity ψ that is much larger than the one found for HR 3831 ($\psi \approx 4$ deg). For each period of oscillation, the maxima of the dipole mode and the magnetic field are in phase when the mode axis crosses the plane formed by the magnetic and rotation axes, i.e. the (X, Z) plane.

the polar axis aligned with the rotation axis, each eigenmode is represented by a single spherical harmonic. One mode is linearly polarized along the rotation axis ($\delta \approx \pi/2, \psi \approx 0$) whereas the two other modes are circularly polarized in two opposite senses ($\delta \approx 0, \psi \approx \pm\pi/4$) in the rotational equatorial plane.

For large values of $|\mu| \gg 1$, the magnetic effects dominate over the rotational ones. In that case, this axisymmetric harmonic ($m = 0$), defined in the system with the polar axis aligned with the magnetic axis, is not coupled with the non-axisymmetric ones ($m = \pm1$). The axisymmetric mode is linearly polarized along the magnetic axis. As shown in Fig. 2 two modes are circularly polarized ($\psi = \pm\pi/4$) for $\beta = 0$ and tend to be linearly polarized as $\beta \to \pi/2$ since the effects of the Coriolis force decrease. For almost perpendicular rotation and magnetic axes, $\beta \approx \pi/2$, the three dipoles modes are linearly polarized along the three orthogonal axes of the magnetic system.

In the intermediate regime, $|\mu| \approx 1$, the situation is more complex. In that case the centrifugal and magnetic effects are comparable which leads to an inclination of the mode system between magnetic and rotation axes.

4.2. Magnetic dipole modes

It is generally believed that modes excited in roAp stars are nearly aligned with the magnetic axis. In principle, as long as $\beta \neq 0$, none of the three dipole modes is strictly aligned with the magnetic field. Let us note that the strict alignment requires $|\delta| = |\pi/2 - \beta|$ and $\psi = 0$. From the three modes, we would like to select the one which is the most aligned with the magnetic axis.

A convenient measure of the departure from strict alignment is the quantity $\mathcal{D}$ defined as follows

$$\mathcal{D} = \frac{\sum_{m\neq0} \alpha_m^2}{\sum_m \alpha_m^2}. \tag{37}$$

For a pure axisymmetric mode ($\alpha_{m\neq0} = 0$) we have $\mathcal{D} = 0$, and in the opposite case for a pure non-axisymmetric mode ($\alpha_0 = 0$) we have $\mathcal{D} = 1$. For dipole modes, with the help of Eqs. (19), (32), (33) and (36), we can express $\mathcal{D}$ in terms of the polarization angles (δ, ψ) defined in the rotation system, as follows

$$\mathcal{D} = 1 - \sin^2(\delta - \beta) \cos^2 \psi. \tag{38}$$

This quantity is plotted in Fig. 2 as function of the obliquity angle β and the magnetic strength parameter μ. For large magnetic fields, $|\mu| > 1$, we see that there is a mode which for all values of β remains linearly polarized along the magnetic axis ($\mathcal{D} \to 0$). At weaker fields ($|\mu| \leq 1$) various modes may approach $\mathcal{D} = 0$ and at the same time have significant values of ψ implying elliptical polarizations. The case of the mode represented by the dashed line is quite interesting for large β. The geometrical picture is that the mode moves the sphere along an elongated ellipse whose major axis is close to the magnetic axis.

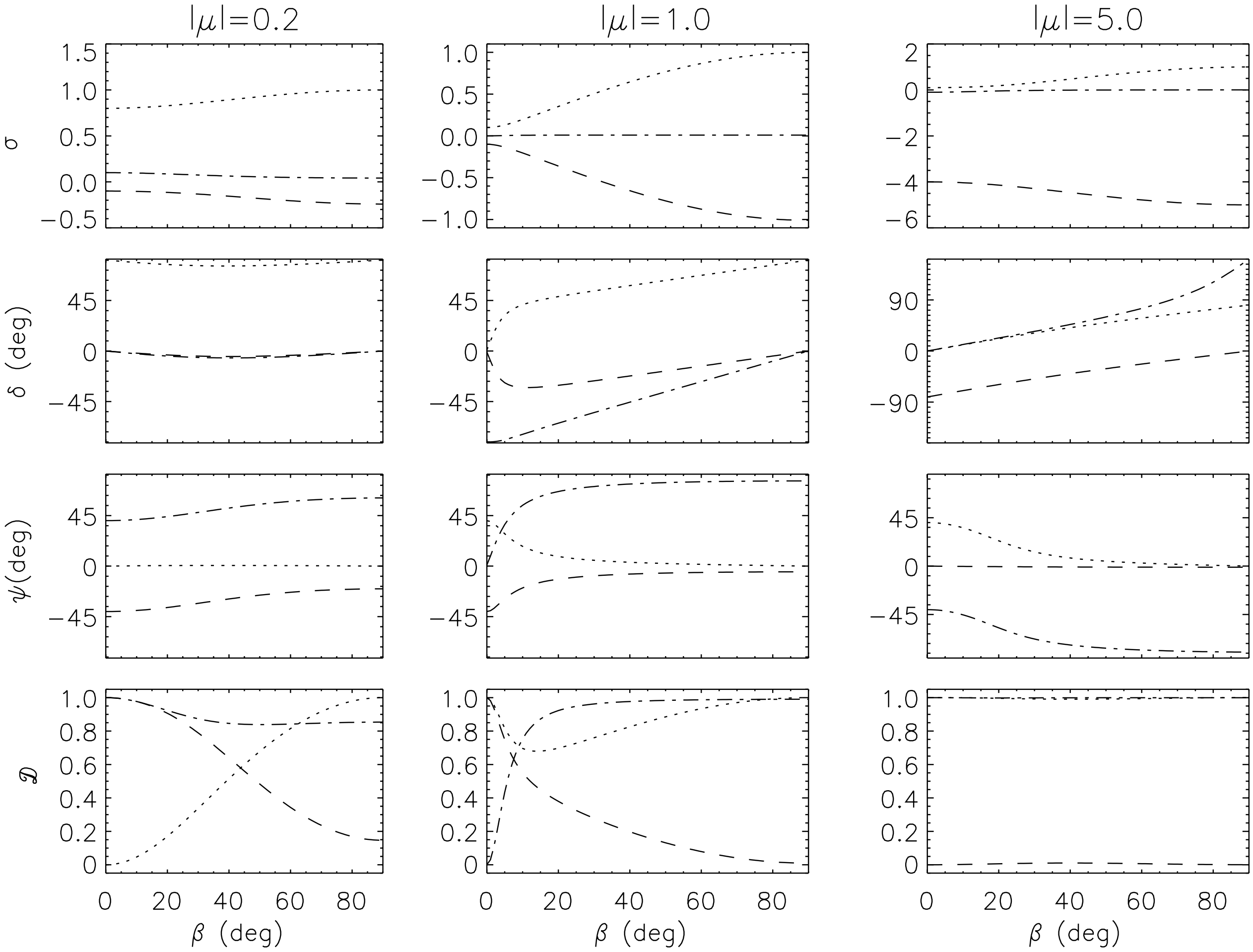

Fig. 2. Plots of the mode parameters for ($\ell = 1$) as functions of the obliquity angle β. The dimensionless frequency σ, the inclination of the mode δ, the polarization angle ψ and the quantity $\mathcal{D}$ are plotted on the four rows of the graph. Each column corresponds to a specific magnetic regime represented by three values of $|\mu|$. The left column corresponds to a rotation dominating regime ($|\mu| = 0.2$) in which two modes (dotted and dashed) can be nearly aligned ($\mathcal{D}$) with the magnetic axis. For a dominating magnetic regime plotted in the right column ($|\mu| = 5$), only one mode (dashed) is aligned with the magnetic axis.

These results show that generally speaking, for reasonable values of μ and for $\beta \neq 0\,[\pi/2]$, none of the three dipole modes is linearly polarized along the magnetic axis ($\mathcal{D} \neq 0$).

4.3. Amplitudes in observer's system

As it follows from Sect. 3, each of the three $\ell = 1$ eigenmodes is seen as a triplet with peaks separated exactly by the rotation rate, Ω. Further, the inequality of the side peaks, $A_{\pm 1}$, arises only from the Coriolis force. From Eqs. (23) and (25) we have

$$\gamma^- = \frac{A_1 - A_{-1}}{A_1 + A_{-1}} = \begin{cases} \frac{\tan\psi}{\cos\delta} & \text{if } |\tan\psi| < \cos\delta \\ \frac{\cos\delta}{\tan\psi} & \text{if } |\tan\psi| > \cos\delta. \end{cases} \qquad (39)$$

The role of the Coriolis force is essential for this inequality since it modifies the ellipticity ψ of the mode which determines the size of γ^-. The sign of γ^- is related to the sense of polarization. Negative values means counter-clockwise elliptical polarization. Note that $|\gamma^-| \leq 1$. The value of $\gamma^- = 1$ corresponds to $A_{-1} = 0$ and $\gamma^- = -1$ to $A_1 = 0$.

This equation emphasizes also the role of centrifugal force in determining the inequality of side peaks through the angle δ.

The sum of the side peak amplitudes to central peak ratio, γ^+, depends on two effects as we have already discussed in Sect. 3. For $\ell = 1$, Eq. (26) becomes

$$\gamma^+ = \frac{A_1 + A_{-1}}{A_0} = \begin{cases} \tan i \cot\delta & \text{if } |\tan\psi| < \cos\delta \\ \tan i \frac{|\tan\psi|}{\sin\delta} & \text{if } |\tan\psi| > \cos\delta. \end{cases} \qquad (40)$$

The value of γ^+ is determined by the inclination of the mode plane to the rotation axis and the inclination of the rotation axis to the line of sight. In Fig. 3 we plot γ^- and $\gamma^+/\tan i$ as functions of β and for the same values of μ used in Fig. 2. Note the rapid decline of $|\gamma^-|$ with β that corresponds to a decreasing effect of the Coriolis force as discussed in Sect. 4.1.

In the strong magnetic field regime, $|\mu| \gg 1$, we find for the mode which is the most aligned with the magnetic field axis ($\mathcal{D} = 0$), the well-known relations for amplitude ratios of the oblique pulsator model (e.g. Kurtz & Shibahashi 1986; Unno et al. 1989),

$$\gamma^- \approx \frac{\chi}{\mu} \to 0 \quad \text{and} \quad \gamma^+ \approx \tan i \tan\beta. \qquad (41)$$

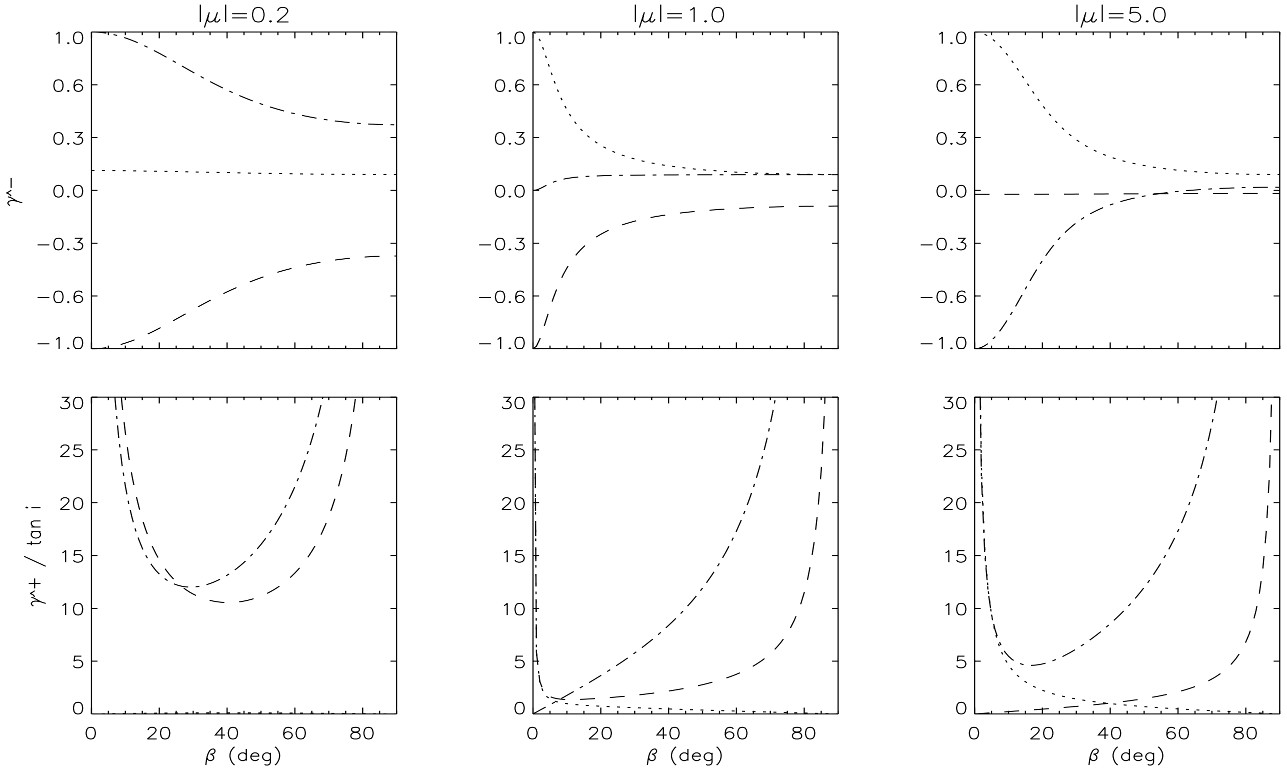

Fig. 3. Plots of the amplitude ratios γ^- (top) and γ^+ (bottom) as functions of the obliquity angle β and for different magnetic regimes $|\mu| = 0.2$ (left), 1.0 (middle) and 5.0 (right). The curves correspond to the three dipole modes. The notation remains the same as in Fig. 2.

In the weak magnetic field regime, $|\mu| \ll 1$, the triplet reduces to a single dominant peak. Indeed, for the mode linearly polarized along the rotation axis ($\delta \approx \pi/2, \psi \approx 0$) we have $A_{\pm 1} \approx 0$ and $A_0 \neq 0$. For the two circularly polarized modes in the rotational equatorial plane ($\delta \approx 0, \psi \approx \pm\pi/4$), we have $A_0 \approx 0$. One of the two amplitudes A_1 or A_{-1} also vanishes depending on the mode that we consider ($\psi = +\pi/4$ or $-\pi/4$). This situation is the worse for asteroseismology since we lose information about the triplet.

Here we see the role of the centrifugal force in the side peak inequality because it is this force that determines the value of $|\mu|$ and then the regime to consider.

5. Application to a representative model of roAp star

We select a model characterized by the following parameters $M/M_\odot = 2.0$, $R/R_\odot = 1.85$, $T_{\rm eff} = 8100$ K, $X_{\rm c} = 0.61$. These parameters are not far from those adopted for HR 3831 – the roAp star with accurately measured unequal side peak amplitudes. The rotation period of this star is $P_{\rm rot} = 2\pi/\Omega \approx 2.85$ d and the central peak frequency of the multiplet is 1.428 mHz. Unfortunately our calculations are not applicable to this object if the polar $B_{\rm p}$ value of 14 kG, as inferred by Bagnulo et al. (1999), is true. The validity of our theory cannot be much extended beyond a 1 kG field. This is illustrated in Fig. 4 which shows in particular that the assumption of single ℓ at fields above 1 kG is incorrect. The μ dependence on $B_{\rm p}$ is complicated and it would be foolish to extrapolate the results by 1 order of magnitude.

Let us stress that the value of $\beta = 8 \pm 1$ deg determined by Bagnulo et al. (1999) is very different from that assumed by Kurtz for modelling light variations in this star (e.g. Kurtz 1992). We also emphasize that it is quite surprising to see any inequality of the side peaks if the mode is aligned with the magnetic axis and if $B_{\rm p}$ is as large as Bagnulo et al. (1999) found since the effects of the Coriolis force would be negligible compared to the magnetic ones.

In view of the controversial data about magnetic field in HR 3831 we have arbitrarily assumed $B_{\rm p} = 1$ kG to calculate μ shown in Fig. 5. Note that our values are always negative. In the same figure we show corresponding values of χ.

At the fastest rotation and the adopted value of $B_{\rm p}$ we are in the $|\mu| < 1$ regime (first column in Fig. 2). As soon as we depart from $\beta = 0$ the alignment of the mode with the magnetic axis is lost. At the lowest rotation, we are in the $|\mu| \gg 1$ regime (third column in Fig. 2). At all values of β one mode is aligned with the magnetic axis, $\mathcal{D} \approx 0$. We may suppose that with the Bagnulo et al. (1999) value of $B_{\rm p}$ we will be in the same regime.

One sees that even at the longest rotational period the value of χ is below 0.1 that we have adopted in Fig. 2. The centrifugal distortion is thus always the dominant rotational effect in determining the orientation of the mode plane, δ.

The two angles that are related to the side peak inequality are ψ and δ. When looking at Figs. 2 and 3 we have to keep in mind that we have used here the value of χ which is at least one order of magnitude larger than the values seen in Fig. 5, which are realistic. Qualitatively the pattern remains the same, however the values of ψ stay close to 0 or $\pm\,\pi/2$ through much wider range of β. Such values of ψ imply linear polarizations.

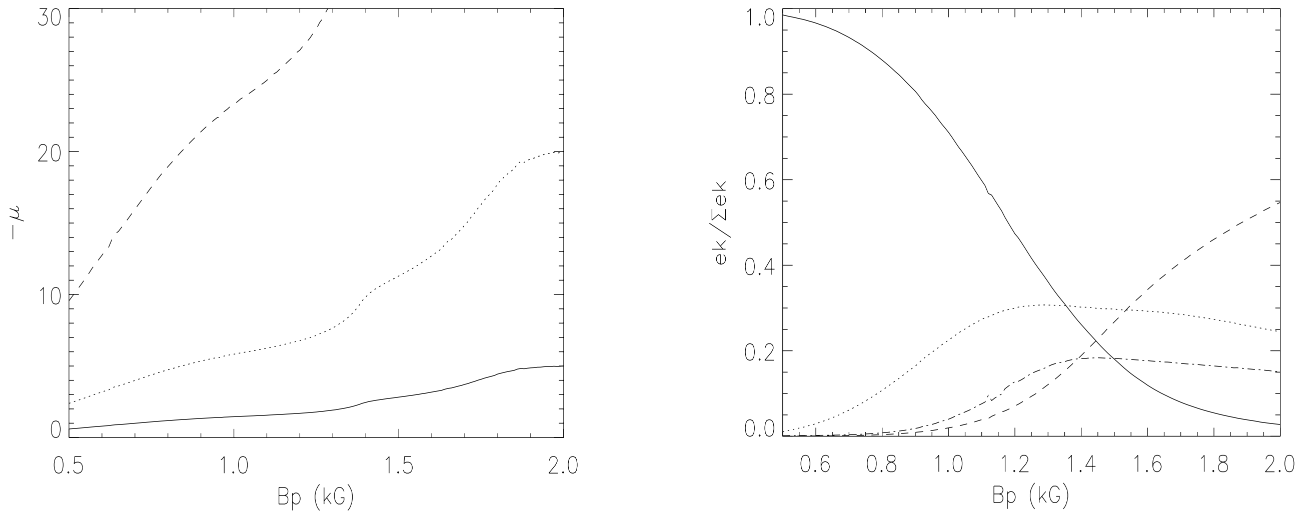

Fig. 4. (Left panel) Plots of the magnetic parameter μ as function of the photospheric magnetic strength $B_{\rm p}$ obtained with the stellar model given in the text, for three different periods of rotation, $P_{\rm rot}$ = 3 d (full line), = 6 d (dotted), = 12 d (dashed). The frequency is 1428 μHz. Note that these ratios are always negative. When $|\mu| > 1$ the magnetic effects dominate over the rotational (centrifugal) effects. (Right panel) Plots of the relative kinetic energies of each ℓ-components of the mode generated by the magnetic field, as functions of $B_{\rm p}$. In all cases $m = 0$, $k = 1$ (full line), $k = 3$ (dashed line), $k = 5$ (dot-dashed line) and $k = 7$ (dotted line), see Bigot et al. (2000) for more details. For $B_{\rm p} < 1$ kG, the $k = 1$ component dominates and the mode is almost a dipole.

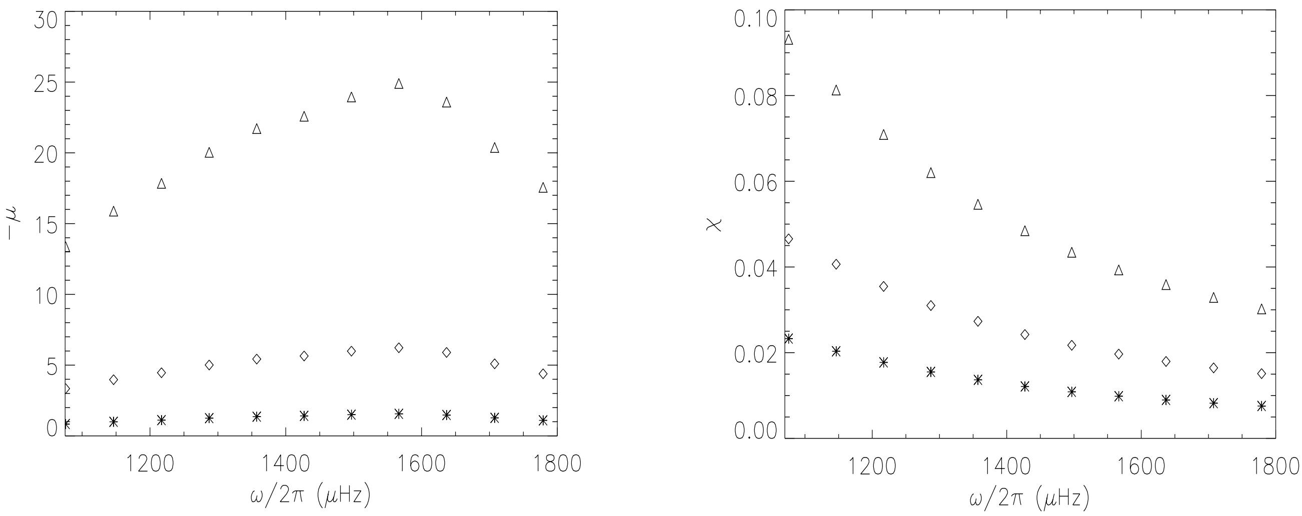

Fig. 5. The values of the parameter μ, see Eq. (30), and χ, see Eq. (29), are plotted against mode frequencies for three values of rotational periods, $P_{\rm rot}$ = 3 d (∗), 6 d (◇), 12 d (△). The value of $B_{\rm p}$ = 1 kG is assumed. The parameters of the adopted Main Sequence star model of 2 $M_\odot$ are given in the text.

In Fig. 6 we plot the amplitudes of the triplet (A_0, A_1, A_{-1}) as functions of the magnetic field. It is clear that the structure of the triplet is strongly affected by the magnetic field's configuration ($B_{\rm p}$,β). In the weak field regime, the triplet reduces to a single peak, since in rotation system each eigenmode is represented by a single spherical harmonics. In the strong field regime, the tendency is to have almost equal side peaks, $A_1 \approx A_{-1}$, for the mode nearly aligned with the magnetic field ($\mathcal{D} = 0$).

For the mode verifying $\mathcal{D} \approx 1$, the inequality of the side peaks increases with the magnetic field strength, $B_{\rm p}$, since as we have already emphasized in Sect. 4.1, the ellipticity of the mode increases.

This inequality of side peaks increases when β decreases since in that case the Coriolis effects are stronger.

For HR 3831, the measured values of the γ's are

$$\gamma^{(-)}_{\rm obs} = 0.097 \pm 0.003 \quad \text{and} \quad \gamma^{(+)}_{\rm obs} = 8.619 \pm 0.187, \qquad (42)$$

(Kurtz et al. 1997). We made an attempt to reproduce the observed values allowing wide ranges of $B_{\rm p}$, β and i values. The value of χ is 0.012. Following the standard assumption, we first considered modes with $\mathcal{D} \approx 0$, i.e. nearly aligned with the magnetic field. None of the combination of the three parameters brought us even close to the observational data. We have succeeded only for a mode which is well inclined to the magnetic axis. In view of the geometrical picture, this dipole mode moves the sphere along a very elongated ellipse

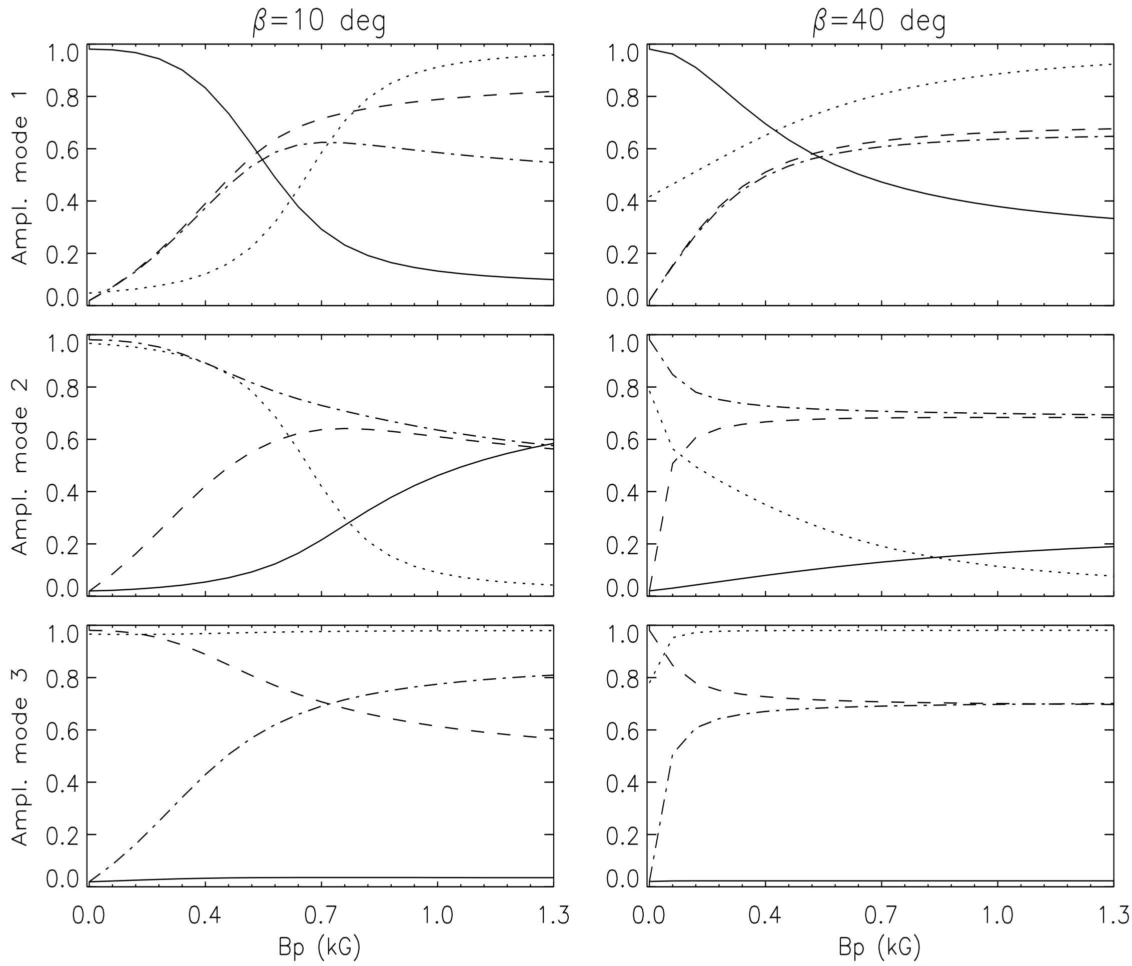

Fig. 6. Plots of the amplitudes A_0 (full line) A_1 (dashed line) and A_{-1} (dot-dashed line) of the triplet as functions of the magnetic field strength B_p and for two values of the obliquity angle β. Each row corresponds to one of the three dipole mode solutions of Eq. (7). The amplitudes are normalized by $(A_0^2 + A_1^2 + A_{-1}^2)^{1/2}$. The dotted line corresponds to the quantity $\mathcal{D}$ that measures the mode inclination with the magnetic field ($\mathcal{D} = 0$ means alignment). The aspect angle is $i = 80$ deg.

($2 < \psi < 5$ deg) in a plane which is very inclined from the magnetic axis ($2 < \delta - \beta < 65$ deg). The range of the parameters leading to γ's consistent with observations are $B_p \in [1-2]$ kG, $\beta \in [4-19]$ deg and $i \in [75-87]$ deg.

These large ranges of parameters are reduced if we fix one of these, e.g. the inclination angle i. If we consider that $i =$ 84 deg, we get more precise values: $\beta = 6.9-7.3$ deg, $\delta - \beta =$ 41.2–41.4 deg, $\psi = 3.60-3.82$ deg and $B_p = 700$ G. We cannot determine them more precisely because of the error bars in the observations of the γ's. They also depend on the stellar model that we consider which is subject to uncertainties.

This inclination of the mode axis is very different from the common interpretation of the oscillations in roAp stars in terms of a mode axis aligned, or nearly aligned, with the magnetic axis. The role of centrifugal force in producing this mode geometry is essential. Only with this force, we have modes which are nearly linearly polarized along an inclined axis in the plane formed by the magnetic and rotation axes.

There are two observational constraints generally accepted to discuss the geometry of the mode in roAp stars. The first one is the phase jump by π radians of the oscillation at the amplitude minimum. This is clearly seen in HR 3831 (e.g. Kurtz et al. 1997). This indicates that the mode is a dipole, or very close to a dipole, say with two hemispheres shifted by π radians, one in contraction and one in expansion. For any inclination of the mode axis, except when the mode is aligned with the rotation axis ($|\delta| = \pi/2$), a phase shift occurs as the star rotates since we see alternatively the two hemispheres.

The second observational fact is the "apparent" coincidence between the times of magnetic and pulsation maxima, e.g. Kurtz et al. (1992) for HR 3831. In fact, for a near coincidence of the envelope of the luminosity-variation curve and magnetic field maxima, it suffices that the pulsation axis stays close to the plane that is formed by the magnetic axis and the rotation axis. This condition is fulfilled with our small value of ψ.

In short, the inclined dipole mode that we found is consistent with these observational constraints.

6. Conclusion and discussion

The aim behind our project was to improve the oblique pulsator model and the hope was that this would solve problems posed by the pulsation data of the roAp star HR 3831. We have shown that the hitherto ignored effect of centrifugal force in modelling pulsations in roAp stars is quite important. In fact, the contribution of the Coriolis force to the total rotational frequency shift is two orders of magnitude less than that of the centrifugal force. Another improvement is the treatment of the magnetic field effects by a non-perturbative approach. However, the adopted treatment is still approximate and needs further improvements.

The observables of interest have been the relative amplitudes of the $(2\ell + 1)$ components of the multiplet in observer's system. We showed that the inequality of side peak amplitudes is determined by the Coriolis force. Only this force can make a difference between prograde and retrograde components of the mode, which is essential for the inequality of amplitudes. Even if the centrifugal force is not responsible for that inequality, at the quantitative level it is very significant.

In greater detail we have discussed the case of $\ell = 1$ modes – the most important ones for modelling pulsations in roAp stars. We have developed a simple geometrical picture for these modes in the presence of rotation and a magnetic field. We have shown that during the pulsation cycle these dipole modes displace the star in general along an ellipse whose orientation in the stellar reference system is determined mostly by the balance between the centrifugal distortion and the magnetic field effects. We also showed that the shape of this ellipse is determined by the Coriolis force. The amplitude ratios in the observer's system are given in terms of the geometrical properties of this ellipse, i.e. orientation and eccentricity.

We did not succeed in solving problems of HR 3831 within the framework of the standard version the oblique pulsator model in which the mode is nearly aligned with the magnetic field. Indeed, we found that the observed mode in HR 3831 is significantly inclined from the magnetic axis; this then is in contradiction with the common idea of aligned magnetic and pulsation axes. Perhaps our failure is due to inadequacies of our treatment of the magnetic field. Still, we would like to point out that we succeeded in reproducing the observed amplitude ratios with this mode inclined to the magnetic field. The maxima of pulsation amplitude for this mode occur close to the plane determined by the rotation and magnetic axes. Such a mode geometry is possible only if effects of centrifugal force are taken into account. The Coriolis force is much weaker and hardly influences the inclination. However, it is responsible for small departure of the maxima from that plane and for the observed inequality of the side-peaks. Such a possibility of a dipole mode inclined with respect to the magnetic axis deserves some consideration in view of the fact that the problem of mode selection in roAp stars is far from being understood. Balmforth et al. (2001) explained preferential excitation of the mode aligned with the magnetic field by invoking an inhibiting effect of the magnetic field on convection and an inhibiting effect of convection on oscillations. Both effects are very difficult to study and this explanation must be regarded only as a possibility.

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Appendix A: The $C_{n\ell}$ and $D_{n\ell}$ coefficients

For our application, we use the standard expression for eigenvectors in a non-rotating and non-magnetic star. These are defined by

$$\boldsymbol{\xi}_{n\ell m} = \xi_{n\ell,r} Y_\ell^m \boldsymbol{e}_r + \xi_{n\ell,h} \boldsymbol{\nabla} Y_\ell^m. \tag{A.1}$$

The radial eigenfunctions, $\xi_{n\ell,r}$ and $\xi_{n\ell,h}$, which in general must be determined only numerically, are m-independent (e.g. Unno et al. 1989).

The Ledoux constant is given by

$$C_{n\ell} = \frac{1}{I_{n\ell}} \int_0^R \left(2\,\xi_{n\ell,r}\,\xi_{n\ell,h} + \xi_{n\ell,h}^2 \right) \rho\, r^2\, \mathrm{d}r, \tag{A.2}$$

where

$$I_{n\ell} = \int_0^R \left(\xi_{n\ell,r}^2 + \Lambda_\ell \xi_{n\ell,h}^2 \right) \rho\, r^2\, \mathrm{d}r. \tag{A.3}$$

At $n \gg 1$ we have $\xi_{n\ell,r}/\xi_{n\ell,h} \propto n^{-1}$. However, we have $C_{n\ell} \propto n^{-2}$ because the contribution from the leading term in $\xi_{n\ell,r}\xi_{n\ell,h}$ vanishes upon integration.

The distortion coefficient has a more complex form but for high radial order p-modes, a good approximation is (Dziembowski & Goode 1992)

$$D_{n\ell} = \frac{4}{3} \frac{\omega_{n\ell}}{I_{n\ell}\omega_{\mathrm{dyn}}^2} \int_0^R \left(\frac{r}{R} \right)^3 \xi_{n\ell,r}^2\, \rho\, r^2\, \mathrm{d}r. \tag{A.4}$$

With $\omega_{n\ell} \propto n$ we roughly have $D_{n\ell} \propto n$.

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