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Images for a binary gravitational lens from a single real algebraic equation

H. Asada[*]

Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan

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Abstract. It is shown that the lens equation for a binary gravitational lens being a set of two coupled real fifth-order algebraic equations (equivalent to a single complex equation of the same order) can be reduced to a single real fifth-order algebraic equation, which provides a much simpler way to study lensing by binary objects.

Key words. gravitational lensing – binaries: general – planetary systems

1. Introduction

The gravitational lensing due to a binary system has attracted a lot of interest since the pioneering work by Schneider & Weiß (1986). The lens equation adopted until now is a set of two coupled real fifth-order equations, equivalently a complex fifth-order equation (Witt 1990) which is based on a complex notation introduced by Bourassa et al. (1973, 1975).

The number of images is classified by curves called caustics, on which the Jacobian of the lens mapping vanishes on a source plane. Caustics for two-point masses are investigated in detail and locations of caustics are clarified based on a set of two coupled real fifth-order equations as an application of catastrophe theory (Erdl & Schneider 1993) and on a complex formalism (Witt & Petters 1993), which is developed as an efficient method to compute microlensed light curves for point sources (Witt 1993): in the binary lensing, three images appear for a source outside the caustic, while five images are caused for a source inside the caustic. For a symmetric binary with two equal masses, the lens equation for a source on the symmetry axes of the binary becomes so simple that we can find the analytic solutions (Schneider & Weiß 1986). In star-planet systems, the mass ratio of the binary is so small that we can find approximate solutions in general (Bozza 1999; Asada 2002). The approximate solutions are used to study the shift of the photocenter position by the astrometric microlensing (Asada 2002).

Nevertheless, it is quite difficult to solve these equations, since there are no well-established methods for solving coupled nonlinear equations numerically with sufficient accuracy (Press et al. 1988). We show that the lens equation can be reduced to a single master equation which is fifth-order in a real variable with real coefficients. As a consequence, the new formalism provides a much simpler way to study the binary lensing.

[*] e-mail: asada@phys.hirosaki-u.ac.jp

2. Lens equation for a binary system

We consider a binary system of two bodies with mass M_1 and M_2 and separation vector $\boldsymbol{L}$ from the object 1 to 2. For a later convenience, let us define the Einstein ring radius angle as

$$\theta_{\mathrm{E}} = \sqrt{\frac{4GMD_{\mathrm{LS}}}{c^2 D_{\mathrm{L}} D_{\mathrm{S}}}}, \tag{1}$$

where G is the gravitational constant, M is the total mass $M_1 + M_2$, and D_{L}, D_{S} and D_{LS} denote distances between the observer and the lens, between the observer and the source, and between the lens and the source, respectively. We choose the position of the object 1 as the coordinate center. In the unit of the Einstein ring radius angle, the lens equation reads

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \left(\nu_1 \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|^2} + \nu_2 \frac{\boldsymbol{\theta} - \boldsymbol{\ell}}{|\boldsymbol{\theta} - \boldsymbol{\ell}|^2}\right), \tag{2}$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ denote the vectors for the position of the source and image, respectively and we defined the mass ratio and the angular separation vector as

$$\nu_1 = \frac{M_1}{M_1 + M_2}, \tag{3}$$

$$\nu_2 = \frac{M_2}{M_1 + M_2}, \tag{4}$$

$$\boldsymbol{\ell} = \frac{\boldsymbol{L}}{D_{\mathrm{L}} \theta_{\mathrm{E}}}. \tag{5}$$

We have an identity $\nu_1 + \nu_2 = 1$. For brevity's sake, ν_2 is denoted by ν. Equation (2) is a set of two coupled real fifth-order equations for (θ_x, θ_y), equivalent to a single complex fifth-order equation for $\theta_x + i\theta_y$ (e.g. Witt 1990, 1993).

Let us introduce polar coordinates whose origin is located at the mass M_1 and the angle is measured from the separation axis of the binary. The coordinates for the source, image and separation vector are denoted by $(\beta_x, \beta_y) = (\rho \cos\varphi, \rho \sin\varphi)$,

$(\theta_x, \theta_y) = (r\cos\phi, r\sin\phi)$ and $(\ell_x, \ell_y) = (\ell, 0)$, respectively, where ρ, r and $\ell \geq 0$. For brevity's sake, $\cos\varphi$ and $\sin\varphi$ are written as C and S, respectively. The parallel and vertical parts of Eq. (2) with respect to the separation vector $\boldsymbol{\ell}$ are

$$r\cos\phi\left(1 - \frac{1-\nu}{r^2} - \frac{\nu}{r^2 - 2\ell r\cos\phi + \ell^2}\right)$$
$$= \rho C - \frac{\nu\ell}{r^2 - 2\ell r\cos\phi + \ell^2}, \tag{6}$$

$$r\sin\phi\left(1 - \frac{1-\nu}{r^2} - \frac{\nu}{r^2 - 2\ell r\cos\phi + \ell^2}\right) = \rho S. \tag{7}$$

First, let us investigate a case of $\rho = 0$, which corresponds to the source being located behind object 1. Then, Eq. (6) becomes

$$r\cos\phi\left(1 - \frac{1-\nu}{r^2} - \frac{\nu}{r^2 - 2\ell r\cos\phi + \ell^2}\right)$$
$$= -\frac{\nu\ell}{r^2 - 2\ell r\cos\phi + \ell^2}, \tag{8}$$

whose right-hand side does not vanish because of $\nu \neq 0$ for the binary. This leads to $\sin\phi = 0$, since the R. H. S. of Eq. (7) does vanish. By using the Cartesian coordinates $(\theta_x, \theta_y) = (r\cos\phi, 0)$ and $\theta_x^2 = r^2$, Eq. (8) is rewritten as

$$\theta_x^3 - \ell\theta_x^2 - \theta_x + \ell(1-\nu) = 0. \tag{9}$$

All solutions for this equation will be given later by Eqs. (31) and (32) as the limit of $\rho \to 0$. In the following, let us assume that $\rho \neq 0$. As for S, there are two cases, off-axis sources ($S \neq 0$) or sources on the symmetry axis ($S = 0$).

2.1. Off-axis sources

Here, we consider the case that S does not vanish. In order to eliminate the common factor in the L. H. S. of Eqs. (6) and (7), we divide Eq. (6) by Eq. (7), whose R. H. S. does not vanish for $S \neq 0$. We obtain

$$r^2 - 2\ell r\cos\phi + \ell^2 = \frac{\nu\ell}{\rho(C - S\cot\phi)}. \tag{10}$$

Substituting this into $r^2 - 2\ell r\cos\phi + \ell^2$ in Eq. (7) gives us

$$\Big[\ell - \rho(C - S\cot\phi)\Big]r^2\sin\phi - \ell\rho S r - \ell(1-\nu)\sin\phi = 0. \tag{11}$$

We eliminate r^2 in this equation by using Eq. (10). Hence, we obtain

$$r = \frac{R_1(\phi)}{R_2(\phi)}, \tag{12}$$

where we defined

$$\begin{aligned} R_1(\phi) &= (\ell^2\rho C - \ell\rho^2 C^2 - \nu\ell + \rho C)\sin^2\phi \\ &\quad -\rho S(\ell^2 - 2\ell\rho C + 1)\sin\phi\cos\phi \\ &\quad -\ell\rho^2 S^2\cos^2\phi, \end{aligned} \tag{13}$$

$$\begin{aligned} R_2(\phi) &= \rho(C\sin\phi - S\cos\phi) \\ &\quad \times\Big[(\ell - \rho C)\sin 2\phi + \rho S\cos 2\phi\Big]. \end{aligned} \tag{14}$$

We can show that $r\cos\phi$ is a function only of $\tan\phi$. Namely, Eq. (12) is rewritten as

$$r\cos\phi = \frac{\tilde{R}_1(\tan\phi)}{\tilde{R}_2(\tan\phi)}, \tag{15}$$

where we defined

$$\begin{aligned} \tilde{R}_1(\tan\phi) &= (\ell^2\rho C - \ell\rho^2 C^2 - \nu\ell + \rho C)\tan^2\phi \\ &\quad -\rho S(\ell^2 - 2\ell\rho C + 1)\tan\phi - \ell\rho^2 S^2, \end{aligned} \tag{16}$$

$$\begin{aligned} \tilde{R}_2(\tan\phi) &= \rho(C\tan\phi - S) \\ &\quad \times\Big[\rho S + 2(\ell - \rho C)\tan\phi - \rho S\tan^2\phi\Big]. \end{aligned} \tag{17}$$

Equation (15) plays a crucial role in this letter; if we find out $\tan\phi$, Eq. (15) gives us the value of $r\cos\phi$. The remaining task is deriving an equation for $\tan\phi$.

Let us substitute Eq. (12) into Eq. (10), so that r can be eliminated. After lengthy but straightforward computations, we obtain an equation for $\tan\phi$

$$\begin{aligned} &(a_5\tan^5\phi + a_4\tan^4\phi + a_3\tan^3\phi + a_2\tan^2\phi \\ &+a_1\tan\phi + a_0)\tan\phi = 0, \end{aligned} \tag{18}$$

where by the frequent use of $C^2 + S^2 = 1$ we defined

$$a_0 = \nu\ell\rho^3 S^3, \tag{19}$$

$$\begin{aligned} a_1 &= \rho^2 S^2 + 2\ell\rho^3 CS^2 - \ell^2(2\rho^2 S^2 - \rho^4 S^2) \\ &\quad -2\ell^3\rho^3 CS^2 + \ell^4\rho^2 S^2 \\ &\quad -\nu(5\ell\rho^3 CS^2 - 4\ell^2\rho^2 S^2), \end{aligned} \tag{20}$$

$$\begin{aligned} a_2 &= -2\rho^2 CS - 4\ell\rho^3(C^2 S - S^3) \\ &\quad +\ell^2(4\rho^2 CS - 2\rho^4 CS) + 4\ell^3\rho^3 C^2 S - 2\ell^4\rho^2 CS \\ &\quad +\nu\Big[\ell(2\rho S + 8\rho^3 C^2 S - 2\rho^3 S^3) \\ &\qquad - 10\ell^2\rho^2 CS + 2\ell^3\rho S\Big], \end{aligned} \tag{21}$$

$$\begin{aligned} a_3 &= \rho^2 + \ell(2\rho^3 C^3 - 10\rho^3 CS^2) \\ &\quad +\ell^2(-2\rho^2 C^2 + 2\rho^2 S^2 + \rho^4) - 2\ell^3\rho^3 C + \ell^4\rho^2 \\ &\quad +\nu\Big[\ell(-2\rho C - 4\rho^3 C^3 + 6\rho^3 CS^2) \\ &\qquad + 6\ell^2\rho^2 C^2 - 2\ell^3\rho C\Big] \\ &\quad +\nu^2\ell^2, \end{aligned} \tag{22}$$

$$\begin{aligned} a_4 &= -2\rho^2 CS + 8\ell\rho^3 C^2 S - \ell^2(4\rho^2 CS + 2\rho^4 CS) \\ &\quad +4\ell^3\rho^3 C^2 S - 2\ell^4\rho^2 CS \\ &\quad +\nu\Big[\ell(2\rho S - 4\rho^3 C^2 S + \rho^3 S^3) \\ &\qquad - 2\ell^2\rho^2 CS + 2\ell^3\rho S\Big], \end{aligned} \tag{23}$$

$$\begin{aligned} a_5 &= \rho^2 C^2 - 2\ell\rho^3 C^3 + \ell^2(2\rho^2 C^2 + \rho^4 C^2) \\ &\quad -2\ell^3\rho^3 C^3 + \ell^4\rho^2 C^2 \\ &\quad +\nu\Big[-\ell(2\rho C + \rho^3 CS^2) + 2\ell^2\rho^2 C^2 - 2\ell^3\rho C\Big] \\ &\quad +\nu^2\ell^2. \end{aligned} \tag{24}$$

It should be noted that all of these coefficients $a_0, \cdots a_5$ are not singular, since they are polynomials in ℓ, ρ, ν, C and S all of which are finite. In the case of nonvanishing ρ and S, Eq. (7) means that $\sin\phi$ does not vanish and consequently neither $\tan\phi$. Hence, Eq. (18) is reduced to the fifth-order equation for $\tan\phi$,

$$\sum_{i=0}^{5} a_i(\tan\phi)^i = 0. \tag{25}$$

As shown by Galois in the 19th century, a fifth-order equation cannot be solved in the algebraic manner (e.g. van der Waerden 1966). Hence, by solving numerically Eq. (25), the image position is obtained as $(\theta_x, \theta_y) = (r\cos\phi, r\cos\phi\tan\phi)$. It is important to consider a relation of Eq. (25) to the treatment of Witt (1993) in which a single complex fifth-order algebraic equation for $z = \theta_x + i\theta_y$ is obtained: When we use a relation $\tan\phi = -i(z-\bar{z})/(z+\bar{z})$, Eq. (25) can be derived also from the complex equation after lengthy manipulations.

For a source inside the caustic, Eq. (25) has five real solutions corresponding to five images, while it has three real and two imaginary solutions when the source is outside the caustic (Witt 1990; Erdl & Schneider 1993; Witt & Petters 1993). This criteria can be re-stated algebraically by the use of the discriminant D_5 for the fifth-order Eq. (25), which takes a rather lengthy form in general, namely 59 terms (e.g. van der Waerden 1966). It is worthwhile to mention that all of real solutions for Eq. (25) must exist between $-K$ and K, where K is the larger one between 1 and $|a_0/a_5| + |a_1/a_5| + |a_2/a_5| + |a_3/a_5| + |a_4/a_5|$. (For instance, see Sect. 66 in van der Waerden 1966).

In numerical computations, it might be difficult to handle extremely large $\tau \equiv \tan\phi$. In such a case, we can separate $|\tau| \in [0,\infty)$ into $|\tau| \in [0,\tau_c)$ and $|\tau| \in [\tau_c,\infty)$. We choose τ_c so that Eq. (25) can be easily solved for $|\tau| \in [0,\tau_c)$. On the other hand, for $|\tau| \in [\tau_c,\infty)$, instead of Eq. (25), we can solve an equation for $\cot\phi \in (-1/\tau_c, 1/\tau_c)$

$$\sum_{i=0}^{5} a_i(\cot\phi)^{5-i} = 0. \tag{26}$$

2.2. Sources on the symmetry axis

Let us consider the case of vanishing S ($C = \pm1$), namely sources on the symmetry axis, for which analytic solutions can be obtained: For a binary with two equal masses, explicit solutions were found by Schneider & Weiß (1986), while no explicit solutions have been given for an arbitrary mass ratio until now. Hence, analytic solutions which are given below are useful for verification of numerical implementations, since numerical solutions for $S \neq 0$ in Sect. 2.1 must approach analytic ones as $S \to 0$.

For $S = 0$, Eq. (7) implies apparently the following three cases

$$r = 0, \tag{27}$$

$$\sin\phi = 0, \tag{28}$$

$$1 - \frac{1-\nu}{r^2} - \frac{\nu}{r^2 - 2\ell r\cos\phi + \ell^2} = 0. \tag{29}$$

The case of $r = 0$ should be discarded, since the left-hand side of Eq. (7) diverges.

Next, let us consider the case of $\sin\phi = 0$. For this purpose, it is convenient to use the Cartesian coordinates $(\theta_x, \theta_y) = (r\cos\phi, 0)$ and $(\beta_x, \beta_y) = (\rho C, 0) \equiv (\bar{\rho}, 0)$ for $\bar{\rho} \in (-\infty, \infty)$. By using $r^2 = \theta_x^2$, Eq. (6) is rewritten as the third-order equation

$$\theta_x^3 - (\ell + \bar{\rho})\theta_x^2 + (\ell\bar{\rho} - 1)\theta_x + \ell(1-\nu) = 0, \tag{30}$$

which coincides with Eq. (9) as $\bar{\rho} \to 0$. Equation (30) is solved explicitly as

$$\theta_x = 2\sqrt{-p}\cos\sigma + \frac{\ell + \bar{\rho}}{3}, \tag{31}$$

with

$$\sigma = \begin{cases} \dfrac{1}{3}\cos^{-1}\left(\dfrac{q}{2p\sqrt{-p}}\right), \\ \dfrac{1}{3}\cos^{-1}\left(\dfrac{q}{2p\sqrt{-p}}\right) + \dfrac{2\pi}{3}, \\ \dfrac{1}{3}\cos^{-1}\left(\dfrac{q}{2p\sqrt{-p}}\right) + \dfrac{4\pi}{3}, \end{cases} \tag{32}$$

where

$$p = -\frac{1}{9}(\ell + \bar{\rho})^2 + \frac{1}{3}(\ell\bar{\rho} - 1), \tag{33}$$

$$q = -\frac{2}{27}(\ell + \bar{\rho})^3 + \frac{1}{3}(\ell + \bar{\rho})(\ell\bar{\rho} - 1) + \ell(1-\nu). \tag{34}$$

Actually, we can show that $p < 0$ and $q^2 + 4p^3 < 0$, which mean these three solutions exist for any source position. They are on-axis solutions, while the remaining two solutions are off-axis solutions, which are present only for sources within the caustics.

Finally, for the case defined by Eq. (29), Eq. (6) leads to

$$\frac{\nu}{r^2 - 2\ell r\cos\phi + \ell^2} = \frac{\bar{\rho}}{\ell}. \tag{35}$$

Replacing $r^2 - 2\ell r\cos\phi + \ell^2$ in Eq. (29) by this equation, we obtain

$$r^2 = \frac{\ell(1-\nu)}{\ell - \bar{\rho}}, \tag{36}$$

which has the positive solution

$$r = \sqrt{\frac{\ell(1-\nu)}{\ell - \bar{\rho}}}, \tag{37}$$

if and only if $\ell > \bar{\rho}$. Substitution of the solution into Eq. (35) leads to

$$\cos\phi = \frac{1}{2}\sqrt{\frac{\ell - \bar{\rho}}{\ell(1-\nu)}}\left(\ell + \frac{\bar{\rho} - \nu\ell}{\bar{\rho}(\ell - \bar{\rho})}\right). \tag{38}$$

A condition $|\cos\phi| \le 1$ is rewritten as

$$\ell^2\bar{\rho}^4 - 2\ell(\ell^2 - 1 + 2\nu)\bar{\rho}^3 + \left[(1-\ell^2)^2 + 6\nu\ell^2\right]\bar{\rho}^2$$
$$-2\ell(1+\ell^2)\nu\bar{\rho} + \nu^2\ell^2 \le 0. \tag{39}$$

Only if this is satisfied for $\bar{\rho} < \ell$, the two images appear at the location given by Eqs. (37) and (38).

3. Conclusion

We have carefully reexamined the lens equation for a binary system in the polar coordinates. As a consequence, we have derived Eq. (25) for $\tan\phi$. After solving the equation, $r\cos\phi$ is determined by Eq. (15). Hence, the image position $(\theta_x, \theta_y) = (r\cos\phi, r\cos\phi\tan\phi)$ can be determined. Our formulation based on the one-dimensional Eq. (25) is significantly useful compared with previous two-dimensional treatments for which there are no well-established numerical methods (Press et al. 1988); the new formulation enables us to study the binary lensing more precisely with saving time and computer resources. For instance, it is effective in rapid and accurate light-curve fitting to microlensing events, in particular due to star-planet systems.

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