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Letter to the Editor

Marginally stable orbits around Maclaurin spheroids and low-mass quark stars

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Abstract. When the eccentricity of a Maclaurin spheroid exceeds a critical value ($e > 0.83458318$), circular orbits in the equatorial plane are unstable for a range of orbital radii outside the stellar surface – this is a purely Newtonian effect related to the oblateness of the star. The orbital frequency in the marginally stable orbit, and all other orbits, around Maclaurin spheroids goes to zero in the limit $e = 1$. The orbital and rotational frequencies in exact relativistic numerical models of rotating, axially symmetric, quark stars of very low mass ($M \leq 0.1\ M_{\odot}$) coincide with those for Maclaurin spheroids. It is impossible to determine the mass of a rapidly rotating quark star by measuring the maximum orbital frequency alone.

Key words. dense matter – equation of state – stars: neutron – stars: binaries: general – X-rays: stars

1. Introduction

Much of what is known about the masses of astronomical objects has been derived from observations of orbital motion and its theory (dating back to Newton 1687), so an accurate knowledge of the laws of orbital motion is of exceptional interest. It seems ironic that only first corrections to orbital motion caused by oblateness have been discussed for rotating stars, even in Newtonian physics, while the formulae for orbital frequency of black holes are known for all allowed angular momenta of the hole (Bardeen et al. 1972).

Interest in the possible range of orbital frequencies around compact bodies has greatly increased after the discovery of millisecond variability (kHz QPOs) in several low-mass X-ray binaries, including X-ray bursters and black hole candidates (for a review see van der Klis 2000). The orbital frequency around spherical bodies is given by the same formula in the Schwarzschild metric as in Newtonian physics, $\Omega = \sqrt{GM/r^3}$, but there is a difference in the allowed range. There is no limit to how high this Keplerian frequency may become in Newtonian physics as the radius of the orbit around an ever smaller gravitating sphere decreases, while in Einstein's theory of gravitation an upper limit to the frequency in stable circular orbits is attained in the marginally stable orbit (of radius $6GM/c^2 \approx 9\,\mathrm{km} \times M/M_{\odot}$ in the Schwarzschild metric). It had been suggested that this property may be used to test general relativity in the strong field-regime around accreting neutron stars, or to measure the stellar mass, by directly comparing the highest frequency manifest in the X-ray flux with relativistic formulae for the orbital frequency in the marginally stable orbit (Kluźniak & Wagoner 1985; Kluźniak et al. 1990), and now several authors have indeed tried to carry out this program for neutron stars (Kaaret et al. 1997; Zhang et al. 1998; Kluźniak 1998), for quark stars (Bulik et al. 1999a,b; Zdunik et al. 2000a,b; Gondek-Rosińska et al. 2001a,b; Datta et al. 2000), and for black holes (Strohmayer 2001).

In this letter we discuss the Newtonian limit of the maximum orbital frequency around uniformly rotating bodies in equilibrium. We compare analytic formulae, derived in Newtonian physics for bodies of constant density, with the results of fully relativistic numerical calculations carried out for quark stars of very low mass.

2. Maclaurin spheroids

In Newtonian gravity, Maclaurin spheroids describe bodies of uniform density in hydrostatic equilibrium. In addition to their theoretical importance, they should be an excellent approximation for planetoids of moderate mass,

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and – as we show below – for quark stars of planetary mass.

As noticed by Thomas Simpson in 1743 and by d'Alembert, the rotational frequency of a Maclaurin spheroid (Fig. 1) approaches zero not only for $e \to 0$, a "spheroid which departs only slightly from a sphere", but also for $e \to 1$, a "highly flattened spheroid":

$$\Omega^2 = 2\pi G\rho(1-e^2)^{1/2}e^{-3} \times \left\{(3-2e^2)\arcsin e - 3e(1-e^2)^{1/2}\right\}, \quad (1)$$

where ρ is constant and $e = (1-b^2/a^2)^{1/2}$ is the eccentricity of the spheroid, with a the major and b the minor axis of the same (Chandrasekhar 1969).

By explicitly evaluating the integrals in the expression for the potential given in Theorem 7, Chapter 3 of Chandrasekhar (1969), we find that the orbital frequencies, $\omega(r)$, also go to zero in the two limits $e \to 0$ and $e \to 1$. Specifically, at the equator,

$$\omega^2(a) = 2\pi G\rho(1-e^2)^{1/2}e^{-3}\left\{\arcsin e - e(1-e^2)^{1/2}\right\}. \quad (2)$$

The corresponding circular orbits (of radius $r = a$) are stable only for eccentricities $e \le e_{\rm s} \equiv 0.83458318$, and the innermost (marginally) stable circular orbit, present in the equatorial plane for all $e > e_{\rm s}$, has frequency

$$\omega^2_{\rm ms} \equiv \omega^2(r_{\rm ms}) = 0.5276189 \times 2\pi G\rho(1-e^2)^{1/2}e^{-3}, \quad (3)$$

also going to zero for $e \to 1$ (Kluźniak 2001). The expressions (2), (3), for the maximum frequency in circular orbits are plotted in Fig. 2, their numerical values for a specific choice of density can be read off Fig. 3.

We stress that these expressions have been derived in Newtonian physics. The marginally stable orbit is present (and stable circular orbits are absent for $r < r_{\rm ms}$) only because the distribution of mass is non-spherical.

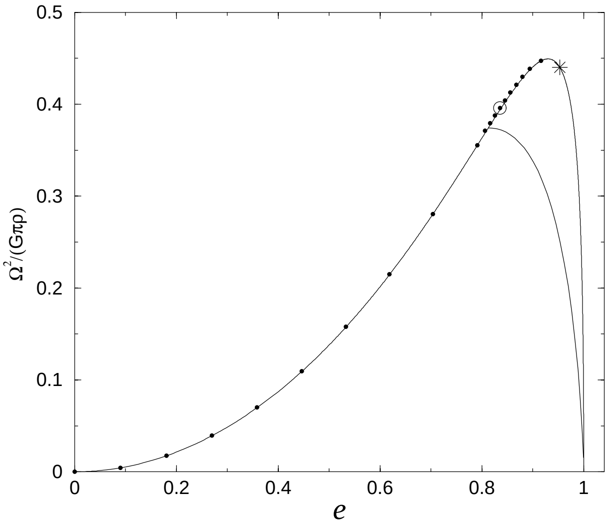

Fig. 1. The rotational frequency of Maclaurin spheroids as a function of their eccentricity (upper curve, the star indicates the point of onset of dynamical instability to a toroidal mode), and the Jacobi sequence (lower curve), after Chandrasekhar (1969). The circle indicates the spheroid for which the marginally stable orbit grazes the equator ($e =$ 0.834583). Also shown (thick dots) are numerical models of uniformly rotating quark stars of mass $M = 0.01\ M_\odot$ calculated by us with the general relativistic spectral code of Gourgoulhon et al. (1999) (the value of "ρ" on the vertical axis used to rescale Ω^2 is the volume averaged density of the non-rotating model).

3. Quark stars with nearly uniform density

Quark stars – if they exist at all – are thought to be formed in (some) supernovae, or in a phase transition in a neutron star exceeding a critical central density, so typically their masses are expected to be comparable to those of neutron stars. However, unlike neutron stars which are unstable below a certain mass, quark stars (or planets) of arbitrarily small mass can exist, on the assumption that quark matter is self bound. It is not impossible that low-mass quark stars may form through fragmentation in a violent event, such as binary coalescence of a more massive quark star and a black hole (Lee & Kluźniak 2001).

Low-mass X-ray binaries are thought to contain stellar mass compact sources, but in most cases the mass has not actually been determined. In a source where much of the luminosity is released in an accretion disk, it would usually be difficult to discriminate directly between a star of mass 1.4 $M_\odot$ with a 10 km radius and, say, a 0.1 $M_\odot$ star of radius 5 km. One indication of a low mass could be a lower photon flux during an X-ray burst, when the luminosity is thought to reach the Eddington limit value (Margon & Ostriker 1973) $L_{\rm Edd} = 9 \times 10^{37}(M/M_\odot)$ erg/s.

Numerical models of static quark stars in general relativity have been constructed by Itoh (1970); Brecher & Caporaso 1976; Witten (1984); Alcock et al. (1986); and others. The first accurate, fully relativistic calculations of rotating quark stars were published by Gourgoulhon et al. (1999), and Stergioulas et al. (1999).

Using the Gourgoulhon code we have computed numerical models of uniformly rotating quark stars in general relativity. We assume quark matter to be self-bound and have used the simplest MIT-bag equation of state $p = (\rho - \rho_0)c^2/3$, to model it. The stars were constrained to be axisymmetric. We have found that for stellar masses much less than that of the sun ($M \ll M_\odot$), the density is nearly uniform throughout the star and, as expected, close to the density ρ_0 at zero pressure of quark matter. We used $\rho_0 = 4.28\times10^{14}$ g/cm^3, and obtained central densities of $4.287, 4.318, 4.479 \times 10^{14}$ g/cm^3, for nonrotating stars with baryon masses of 0.001, 0.01, 0.1 $M_\odot$, respectively, and $4.283, 4.301, 4.399 \times 10^{14}$ g/cm^3 for models of the same baryon masses but rotating at a rate of 1 kHz.

We have found that for $M \le 0.1\ M_\odot$ the Maclaurin spheroid is a very good approximation to the quark star, at least in terms of gross stellar properties (Figs. 1–3). In the figures, we show (as dots) the results of our general-relativistic numerical computations for stars of baryon mass 0.01 $M_\odot$. For such stars, the maximum orbital

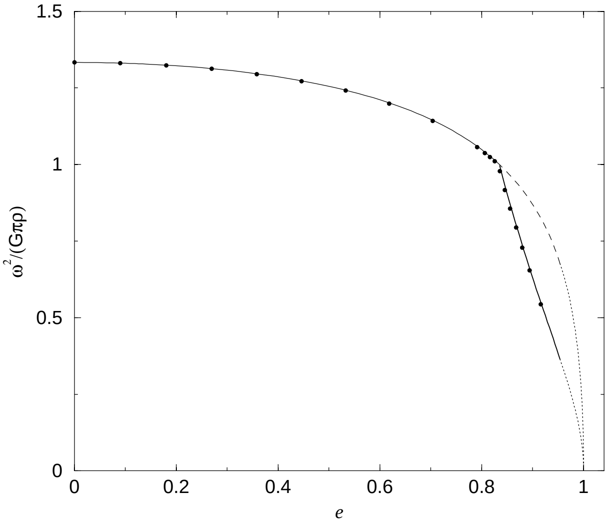

Fig. 2. The maximum frequency in stable circular orbits around the Maclaurin spheroids as a function of the spheroid eccentricity (continuous curves). The thin curve is the orbital frequency at the equator, Eq. (2), the dashed portion corresponds to unstable orbits at the equator. The thick curve is the orbital frequency in the marginally stable orbit, Eq. (3). The dotted portions of the curves correspond to spheroid eccentricities past the point of onset of dynamical instability, at $e = 0.95289$. Note that the curves are Newtonian, the speed of light does not enter Eqs. (2), (3). Also shown (as thick dots) are the numerical models of quark stars presented in Fig. 1.

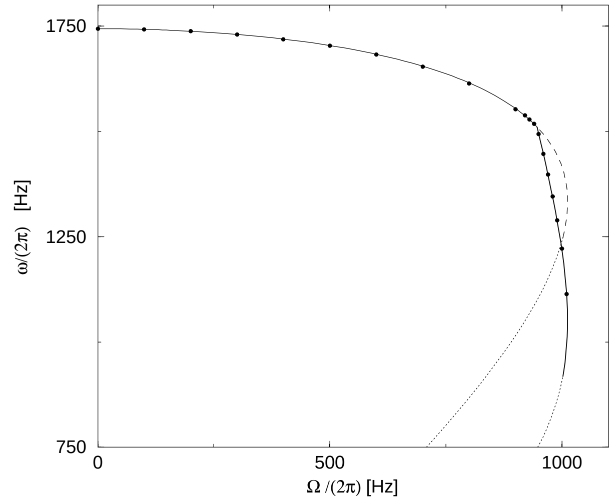

Fig. 3. The maximum orbital frequency as a function of the rotational frequency of the Maclaurin spheroid. All frequencies scale with the square root of the density. The numerical values shown correspond to $\rho = 4.28 \times 10^{14}$ g/cm^3, at this value of the density the marginally stable orbit is present for stellar rotational frequencies $\Omega/(2\pi) > 945$ Hz. All symbols have the same meaning as in Fig. 2, in particular, the steeply descending thick curve corresponds to the marginally stable orbit.

frequency in stable circular orbits (Fig. 3) is attained for the static model ($e = 0$, $\Omega = 0$), and is clearly given by the Keplerian expression $\omega(a) = \sqrt{4\pi G\bar{\rho}/3}$; we have used $\sqrt{3/4}$ times this value to rescale in Figs. 1, 2 the frequencies of all our numerical models. Here $\bar{\rho}$ is the volume averaged density of the static model. For every $e \le e_s$ the maximum stable orbital frequency is attained in a circular orbit on the equator. For larger eccentricities the maximum value of orbital frequency is reached in the innermost stable orbit and it drops precipitously as $e \to 1$. The numerical results are seen to agree with the Newtonian expressions of Eqs. (2), (3). Clearly, general-relativistic effects are negligible in the external metric of low-mass quark stars.

Because in Fig. 3 the dependence of ω on the stellar rotational frequency $(2\pi)^{-1}\Omega$ is shown, the curves are "double valued," for a given rotational frequency the larger value of orbital frequency corresponds to a spheroid/star of lower eccentricity, the smaller to the one for larger e (cf., Fig. 1). In Fig. 3 the intersection of the curves for $\omega_{\rm ms}$ and for $\omega(a)$ on the unstable branch (dashed curve) is a mirage caused by projection onto the Ω axis, in fact, values of about 1250 Hz are attained for different values of e in the case of $(2\pi)^{-1}\ \omega_{\rm ms}$ and of $(2\pi)^{-1}\ \omega(a)$. Compare Fig. 2, where it is clear that $\omega(a)$ and $\omega(r_{\rm ms})$ intersect only for $e = e_s$ and $e = 1$. Dynamical instability to a toroidal deformation sets in at $e = 0.95289$ in the Newtonian theory (Chandrasekhar 1969), for this reason we do not extend the curves in Fig. 3 to low orbital frequencies.

Finally, in Fig. 1, defining the eccentricity as $e = (1 - r_{\rm p}^2/a^2)^{1/2}$, where $r_{\rm p}$ is the polar coordinate radius of the star and a is the r coordinate of the equator, we plot the rotational frequencies of the numerically computed stars, superimposed on Maclaurin's and Jacobi's sequence (Chandrasekhar 1969). We have found that even for higher mass stars (e.g., up to 1.5 $M_\odot$) the rotational frequency in our numerical models departs by no more than several percent from the curve of Fig. 1, up to its maximum.

The circle in Fig. 1 marks the appearance of the marginally stable orbit at $e = e_s$. This occurs above the bifurcation point (at eccentricity $e = e_{\rm J} = 0.8127$) past which the Maclaurin spheroids are secularly unstable to deformation into a Jacobi ellipsoid. However, this does not necessarily imply that the Newtonian marginally stable orbit, discussed here for the Maclaurin spheroid, is irrelevant to quark stars or other stars.

First, a secular instability can grow only in the presence of dissipative mechanisms like viscosity (Roberts & Stewartson 1963) or gravitational radiation (Chandrasekhar 1970; Friedman & Schutz 1978; Friedman 1978). "By a suitable choice of the strength of viscosity relative to gravitational radiation, it is possible to stabilize the Maclaurin sequence all the way up to the point of dynamical instability" (Shapiro & Teukolsky 1983).

Second, for a possibly realistic range of masses of compact stars, effects of general relativity may delay the onset of (viscosity driven) secular instability to non-axisymmetric deformations past the eccentricity where the Newtonian marginally stable orbit (related to rotation-induced stellar flattening) appears. What is the minimum mass of a quark star for which $e_{\rm J} > e_s$? Calculations of Shapiro & Zane (1998) indicate that the bifurcation point occurs at $e_J = 0.835$ for $GM/(Rc^2) = 0.013$, where R is the radius of the non-rotating star of gravitational mass M. For the quark stars presented here, this would occur already for a star with $M = 0.03\ M_\odot$, for which $R = 3.2$ km. It would appear then, that in the mass range $\sim(0.03\ M_\odot, 0.1\ M_\odot)$ quark stars are well approximated by the Maclaurin spheroids *and* post-Newtonian effects may stabilize the quark star at eccentricities sufficiently high that the purely Newtonian marginally stable orbit is present outside the stellar surface, for $e > e_s$.

4. Discussion

The expressions (2), (3), for the orbital frequencies around Maclaurin spheroids, and the presence of the marginally stable orbit in their Newtonian theory, are interesting in their own right. The effect on orbital frequency of quadrupole and octupole moments of mass distribution has been pointed out by Sibgatulin & Sunyaev (1998) and by Shibata & Sasaki (1999); but the Newtonian origin of this phenomenon apparently went unrecognized. The marginally stable orbit in low-mass quark stars was first discovered numerically (Kluźniak et al. 2000), and its Newtonian character was stressed by Zdunik & Gourgoulhon (2000).

It is interesting to note that the gravity of Maclaurin spheroids provides a consistent Newtonian framework in which the influence of the marginally stable orbit on accretion flow can be investigated, e.g., in three dimensional explorations of the hydrodynamics of coalescing binaries.

The kHz QPO phenomenon has been interpreted as giving support to the presence of the general-relativistic marginally stable orbit in accreting neutron stars, and stellar masses $\approx 2\ M_\odot$ have been inferred. But we saw that a maximum orbital frequency of 1 kHz can also be attained in a Newtonian marginally stable orbit for a rapidly rotating star of rather low mass (e.g., $0.1\ M_\odot$). We conclude that it would be premature to assign a mass value to a compact star based on observations of maximal orbital frequency alone, even if there is no doubt that the frequency is obtained in the marginally stable orbit. The detection itself of a marginally stable orbit could be taken to indicate strong-field effects of general relativity only if the star were known to rotate slowly, i.e., at periods $\gg 1$ ms for mean stellar densities exceeding 4×10^{14} g/cm^3.

Low-mass quark stars are well described by Maclaurin spheroids. This is the only known example of an accurate analytic description of a compact stellar remnant composed of matter at supranuclear density. We expect that in general, qualitative features of the exterior gravitational potential of Maclaurin spheroids may be relevant to a discussion of rapidly rotating stars. In black holes frame dragging draws the marginally stable orbit in towards the axis of rotation, as the black hole spin increases, but it is known that for rapidly rotating massive quark stars rotation acts in the opposite sense: the marginally stable orbit is pushed out (Stergioulas et al. 1999). Hints of the same effect are observed in neutron star models at highest masses. We believe that the origin of this behavior is Newtonian, and point out that effects of stellar flattening, present already in Newtonian theory, counteract relativistic effects of frame dragging.

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