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Cosmic ray transport in anisotropic magnetohydrodynamic turbulence

I. Fast magnetosonic waves

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Abstract. Observations of interstellar scintillations, general theoretical considerations and comparison of interstellar radiative cooling in HII-regions, and in the diffuse interstellar medium, with linear Landau damping estimates for fast-mode decay, all strongly imply that the power spectrum of fast-mode wave turbulence in the interstellar medium must be highly anisotropic. It is not clear from the observations whether the turbulence spectrum is oriented mainly parallel or mainly perpendicular to the ambient magnetic field, either will satisfy the needs of balancing wave damping energy input against radiative cooling. This anisotropy must be included when transport of high energy cosmic rays in the Galaxy is discussed. Here we evaluate the relevant cosmic ray transport parameters in the presence of anisotropic wave turbulence. Using the estimates of the anisotropy parameter in the strongly parallel and perpendicular regimes, based on linear Landau damping balancing radiative loss in the diffuse interstellar medium, we show that in nearly all situations the pitch-angle scattering of relativistic cosmic rays by fast magnetosonic waves at pitch-angle cosines $|\mu| \geq V_{\rm A}/c$ is dominated by the transit-time damping interaction. The momentum diffusion coefficient of cosmic ray particles is calculated by averaging the respective Fokker-Planck coefficient over the particle pitch-angle for the relevant anisotropy parameters within values of $10^{-8} \leq \Lambda \leq 10^{11}$. For strongly perpendicular turbulence ($\Lambda \ll 1$) the cosmic ray momentum diffusion coefficient is enhanced with respect to the case of isotropic ($\Lambda = 1$) turbulence by the large factor $\Lambda^{-1/2}$. For strongly parallel turbulence ($\Lambda \gg 1$) the momentum diffusion coefficient is reduced with respect to isotropic turbulence by the large factor $2\Lambda^{s/2}/s$.

Key words. magnetohydrodynamics (MHD) – plasmas – turbulence – cosmic rays – ISM: magnetic fields

1. Introduction

Observations of interstellar scintillations (Rickett 1990; Spangler 1991), general theoretical considerations (Goldreich & Sridhar 1995), and comparison of interstellar radiative cooling in HII-regions and in the diffuse interstellar medium with linear Landau damping estimates for fast-mode decay (Lerche & Schlickeiser 2001), all strongly imply that the power spectrum of fast-mode wave turbulence in the interstellar medium must be highly anisotropic. It is not clear from the observations whether the turbulence spectrum is oriented mainly parallel or mainly perpendicular to the ambient magnetic field, either will satisfy the needs of balancing wave damping energy input against radiative cooling. Theoretically, Goldreich & Sridhar (1995) prefer turbulence organized in ribbon-like stuctures paralleling the ambient field. But, whichever way the turbulence is organized (and one expects that observations over the next decade or so should resolve the current ambiguity), there is little question that it is highly anisotropic.

This anisotropy must be included when transport of high energy cosmic rays in the Galaxy is discussed. So far, with the noteworthy exception of Jaekel & Schlickeiser (1992), in all the literature concerning the determination of cosmic ray transport parameters, there appears to be consideration given only to turbulence which has a power spectrum either slab-like along the ordered magnetic field or isotropically distributed in wavenumber (e.g., Schlickeiser & Miller 1998, hereafter referred to as SM). The purpose of the present paper is to remedy this defect to some extent by evaluating the relevant cosmic ray

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transport parameters in the presence of anisotropic wave turbulence.

2. Turbulence spectrum

A synthesis of current observations would indicate that a plasma wave power spectrum of the form

$$I(\boldsymbol{k}) = I_0\,[k_\parallel^2 + \Lambda k_\perp^2]^{-(2+s)/2} \tag{1}$$

satisfies the needs of the interstellar scintillation observations, the balance of wave energy dissipation and radiative cooling in HII-regions and in the diffuse interstellar medium, and is in accord with the general theoretical arguments advanced by Goldreich & Sridhar (1995). According to Rickett (1990) and Spangler (1991) Eq. (1) is valid for $|\boldsymbol{k}|(\equiv (k_\parallel^2 + k_\perp^2)^{1/2})$ larger than a minimum wavenumber, $k_{\rm min}$, and less than a maximum $k_{\rm max}$. Spangler (1991) identifies these wavenumbers as due to an inner scale length, $l_{\rm min}(\equiv 2\pi/k_{\rm max})$, and an outer scale length $l_{\rm max}(\equiv 2\pi/k_{\rm min})$. Observations indicate that the power spectral index, s, is around 5/3, while normalization of the power spectrum requires

$$(\delta B)^2 = \int {\rm d}^3k\, I(\boldsymbol{k}) = 2\pi I_0 \times \int_{-1}^{1} {\rm d}\eta[\eta^2 + \Lambda(1-\eta^2)]^{-(2+s)/2} \int_{k_{\rm min}}^{k_{\rm max}} {\rm d}k\, k^{-s} \tag{2}$$

where $k_\parallel = k\eta$, $k_\perp = k(1-\eta^2)^{1/2}$, with η being the cosine of the propagation angle of a plasma wave with respect to the ambient magnetic field. Moreover, $(\delta B)^2$ is the fluctuation strenth in the magnetic field, and the constant Λ accounts for the turbulence anisotropy.

Note that if the turbulence is isotropic$(\Lambda = 1)$ then

$$I_0(\Lambda = 1) = \frac{(\delta B)^2}{4\pi} / \int_{k_{\rm min}}^{k_{\rm max}} {\rm d}k\, k^{-s} \tag{3}$$

while for non-isotropic turbulence

$$I_0(\Lambda) = I_0(\Lambda = 1)/J(\Lambda) \tag{4}$$

with the integral

$$J(\Lambda) \equiv \int_0^1 {\rm d}\eta[\eta^2 + \Lambda(1-\eta^2)]^{-(2+s)/2} = {}_2F_1(1+\frac{s}{2}, 1; \frac{3}{2}; 1-\Lambda) \tag{5}$$

which can be expressed in terms of the hypergeometric function.

3. Cosmic ray Fokker-Planck coefficients

On the basis of quasilinear transport theory the general form of the Fokker-Planck coefficients has been given by SM for cosmic ray particles with speeds $v \gg V_{\rm A}$, where $V_{\rm A} = B_0/\sqrt{4\pi\rho}$ is the Alfvén speed in terms of the ambient magnetic field strength, B_0, and the ionized mass density, ρ. Equations (17)–(19) of SM are the relevant factors to examine, representing the Fokker-Planck-coefficients $D_{\mu\mu}$, $D_{p\mu}$ and D_{pp}. Here $\mu = p_\parallel/p$ is the cosine of the pitch angle of a cosmic ray particle of total momentum p.

Quasilinear transport equations for magnetohydrodynamic plasma waves were formulated originally by Kennel & Engelmann (1966), Hall & Sturrock (1967) and Lerche (1968). The quasilinear approach to the interaction of energetic charged particles with partially random electromagnetic fields $(B_0 + \delta B, \delta E)$ is a first-order perturbation calculation in the ratio $q_L = (\delta B/B_0)^2$ and requires smallness of this ratio with respect to unity. In most cosmic plasmas this requirement is well satisfied as has been established by direct in-situ measurements in interplanetary plasmas, or due to saturation effects in the growth of fluctuating fields. Comparison with Monte Carlo simulations of the transport of charged particles with different plasma wave fields (e.g., Michalek & Ostrowski 1996) demonstrates that the quasilinear theory provides an accurate description of cosmic ray transport for ratios $q_L \le 2$.

Due to the high conductivity of most cosmic plasmas, large-scale steady electric fields are absent, so that the interest concentrates on magnetized plasma. By linear stability calculations it has been established that these systems contain low-frequency magnetohydrodynamic turbulence such as shear Alfvén waves and fast and slow magnetosonic waves. For these plasma waves the magnetic part of the Lorentz force is much larger than the electric part of the Lorentz force, so that the time scale for rapid pitch angle scattering of energetic charged particles is much shorter than the time scale for energy changes. In this case the particle's gyrotropic distribution function adjusts rapidly to quasi-equilibrium, which is close to the isotropic distribution function, in excellent agreement with the observational fact of the isotropy of cosmic ray particles. For nonrelativistic $(u \ll c)$ bulk speed of the turbulence-carrying background plasma the diffusion-convection transport equation for the isotropic part of the phase space density $F(z, p, t)$ can be derived by a well-known approximation scheme (Jokipii 1966; Hasselmann & Wibberenz 1968; Earl 1973; Schlickeiser 1989) from the quasilinear Fokker-Planck equation

$$\frac{\partial F}{\partial t} - S_0 = \frac{\partial}{\partial z}\left[\kappa\frac{\partial F}{\partial z}\right] - V\frac{\partial F}{\partial z} + \frac{p}{3}\frac{\partial V}{\partial z}\frac{\partial F}{\partial p} + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 A\frac{\partial F}{\partial p} - p^2\dot{p}_{\rm Loss}F\right] - \frac{F}{T_{\rm c}} \tag{6}$$

where the spatial diffusion coefficient κ, the cosmic ray bulk speed V and the momentum diffusion coefficient A are determined by pitch-angle averages of three Fokker-Planck coefficients

$$\kappa = \frac{v^2}{8}\int_{-1}^{1} {\rm d}\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)} \tag{7}$$

$$V = u + \frac{1}{3p^2}\frac{\partial}{\partial p}(p^3 D),\; D = \frac{3v}{4p}\int_{-1}^{1} {\rm d}\mu (1-\mu^2)\frac{D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)} \tag{8}$$

$$A = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu \left[D_{pp}(\mu) - \frac{D^2_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}\right] . \tag{9}$$

In Eq. (6) S_0 is the source term, and $\dot{p}_{\rm loss}$ and $T_{\rm c}$ describe continous and catastrophic momentum loss processes.

The three Fokker-Planck coefficients, describing particle-wave interaction processes and entering Eqs. (7), (8), (9) are calculated (Hall & Sturrock 1967; Krommes 1984; Achatz et al. 1991) from ensemble-averaged first-order corrections to the particle orbit. Therefore they depend on the tensor components of the plasma wave power spectrum $<\delta B_{\rm l}(\boldsymbol{k})\delta B_{\rm m}(\boldsymbol{k})>$. For a magnetic turbulence tensor with no preferred direction, Batchelor (1953) notes that $<\delta B_{\rm l}(\boldsymbol{k})\delta B_{\rm m}(\boldsymbol{k})>$ can be written in the general form

$$<\delta B_{\rm l}(\boldsymbol{k})\delta B_{\rm m}(\boldsymbol{k})> = \frac{G(\boldsymbol{k})}{8\pi k^2}[\delta_{\rm lm} - \frac{k_{\rm l}k_{\rm m}}{k^2}] + \imath\frac{H(\boldsymbol{k})}{8\pi k^2}\epsilon_{{\rm lm}k}k_k . \tag{10}$$

Application of Cramer's theorem requires $G(k) \geq 0$ for all $\boldsymbol{k}$, and $-G(\boldsymbol{k}) \leq kH(\boldsymbol{k}) \leq G(\boldsymbol{k})$ for all $\boldsymbol{k}$. Then

$$(\delta B)^2 = \int \mathrm{d}^3k \frac{G(\boldsymbol{k})}{4\pi k^2} = \frac{1}{2}\int \mathrm{d}k \int_{-1}^{1} \mathrm{d}\eta G(k,\eta) \tag{11}$$

where $\eta = \boldsymbol{k}\cdot B_0/|\boldsymbol{k}||B_0|$.

For fast-mode waves propagating either forward (phase velocity $\omega/k = jV_{\rm A}$, $j = +1$) or backward (phase velocity $\omega/k = jV_{\rm A}$, $j = -1$) to the ambient magnetic field an index j is used to track the wave direction (SM) and, in principle, the magnetic helicity $H(\boldsymbol{k})$ can also be included in the evaluation of the Fokker-Planck coefficients. However, little is known about any magnetic helicity term in the interstellar turbulence so, in this first investigation of the effects of wave turbulence anisotropy on the cosmic ray transport parameters, we restrict our attention to the anisotropy factor $G(\boldsymbol{k})/(8\pi k^2)$.

With the identification

$$\frac{G(\boldsymbol{k})}{8\pi k^2} = \frac{I_0 k^{-(2+s)}}{[\eta^2 + \Lambda(1-\eta^2)]^{(2+s)/2}} \tag{12}$$

it follows that the anisotropic variants of Eqs. (27)–(29) of SM take the form

$$\begin{aligned} D_{\mu\mu} \;=\; & \frac{2\pi^2\Omega^2(1-\mu^2)}{B_0^2}\sum_{j=\pm1} I_0^j \sum_{n=-\infty}^{\infty} \\ & \times \int_{-1}^{1} \mathrm{d}\eta (1+\eta^2)[\eta^2 + \Lambda(1-\eta^2)]^{-(2+s)/2} \\ & \times \int_{k_{\rm min}}^{k_{\rm max}} \mathrm{d}k\, k^{-s}\delta[kv\mu\eta - jV_{\rm A}k + n\Omega] \\ & \times \left(J_n'\left(\frac{kv(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\Omega|}\right)\right)^2 \end{aligned} \tag{13}$$

$$D_{pp} = \frac{p^2V_{\rm A}^2}{v^2} D_{\mu\mu} \tag{14}$$

$$\begin{aligned} D_{\mu p} \;=\; & \frac{4\pi^2\Omega^2 pV_{\rm A}}{vB_0^2}\sum_{j=\pm1} jI_0^j \sum_{n=-\infty}^{\infty} \\ & \times \int_{-1}^{1} \mathrm{d}\eta [\eta^2 + \Lambda(1-\eta^2)]^{-(2+s)/2} \\ & \times \int_{k_{\rm min}}^{k_{\rm max}} \mathrm{d}k\, k^{-s}\delta[kv\mu\eta - jV_{\rm A}k + n\Omega]\frac{n|\Omega|}{kv} \\ & \times J_n^2\left(\frac{kv(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\Omega|}\right)\left[\frac{\mu}{2} - \frac{n|\Omega|\eta}{kv(1-\eta^2)}\right] \end{aligned} \tag{15}$$

where I_0^j reflects the two intensity components of turbulence forward and backward to the ambient magnetic field, and we have taken both to have the same spectral shape to be in accord with observations. Then $I_0^+ + I_0^- = I_0$, where I_0 is given by Eq. (4).

The general Fokker-Planck coefficients represented through Eqs. (13)–(15) can be split into two parts: components with $n = 0$ (customarily referred to as transit-time contributions), and components with $n \neq 0$ (customarily referred to as gyroresonant contributions). We consider each in turn.

3.1. Transit-time contributions ($n = 0$)

In this case the argument of the δ-functions is just $\delta[k(v\mu\eta - jV_{\rm A})] = k^{-1}\delta[v\mu\eta - jV_{\rm A}]$ so that one has

$$D^T_{\mu p} = 0 \tag{16}$$

$$\begin{aligned} D^T_{\mu\mu} \;=\; & \frac{2\pi^2\Omega^2(1-\mu^2)I_0(\Lambda)}{v|\mu|B_0^2} H[|\mu - \epsilon]\left(1 + \frac{\epsilon^2}{\mu^2}\right) \\ & \times \left[\frac{\epsilon^2}{\mu^2} + \Lambda\left(1 - \frac{\epsilon^2}{\mu^2}\right)\right]^{-(2+s)/2} \\ & \times \int_{k_{\rm min}}^{k_{\rm max}} \mathrm{d}k\, k^{-(1+s)} J_1^2\left(\frac{kv}{|\Omega|}\sqrt{(1-\mu^2)\left(1 - \frac{\epsilon^2}{\mu^2}\right)}\right) \end{aligned} \tag{17}$$

$$D^T_{pp} = p^2\epsilon^2 D^T_{\mu\mu} \tag{18}$$

where $\epsilon = V_{\rm A}/v$ and where $H[x]$ denotes the Heaviside step function. The superscript T refers to the transit-time component.

Using Eqs. (3) and (4) for $I_0(\Lambda)$ Eq. (17) can be written

$$\begin{aligned} D^T_{\mu\mu} \;=\; & \frac{\pi}{2}\left(\frac{\delta B}{B_0}\right)^2 \frac{\Omega^2(1-\mu^2)}{v|\mu|} H[|\mu - \epsilon]\left(1 + \frac{\epsilon^2}{\mu^2}\right) \\ & \times \left[\frac{\epsilon^2}{\mu^2} + \Lambda\left(1 - \frac{\epsilon^2}{\mu^2}\right)\right]^{-(2+s)/2}\left[\int_{k_{\rm min}}^{k_{\rm max}} \mathrm{d}k\, k^{-s}\right]^{-1} \\ & \times \left(\int_0^1 \mathrm{d}\eta [\eta^2 + \Lambda(1-\eta^2)]^{-(2+s)/2}\right)^{-1} \\ & \times \int_{k_{\rm min}}^{k_{\rm max}} \mathrm{d}k\, k^{-(1+s)} J_1^2\left(\frac{kv}{|\Omega|}\sqrt{(1-\mu^2)\left(1 - \frac{\epsilon^2}{\mu^2}\right)}\right) . \end{aligned} \tag{19}$$

Relative to the isotropic ($\Lambda = 1$) situation one can write

$$D^T_{\mu\mu}(\Lambda) = D^T_{\mu\mu}(\Lambda=1)\left[\frac{\epsilon^2}{\mu^2}+\Lambda\left(1-\frac{\epsilon^2}{\mu^2}\right)\right]^{-(2+s)/2} \times\left(\int_0^1 \mathrm{d}\eta[\eta^2+\Lambda(1-\eta^2)]^{-(2+s)/2}\right)^{-1} \quad (20)$$

$$D^T_{pp}(\Lambda) = p^2\epsilon^2 D^T_{\mu\mu}(\Lambda). \quad (21)$$

3.2. Gyroresonance contributions ($n \neq 0$)

In this case the contributions are more complex, as also noted in the isotropic case by SM, due to the fact that the argument of the δ-function now involves the wavenumber k explicitly. Following the same sense of argument as given by SM, after some algebra, one can write the gyroresonance contributions as

$$D^G_{\mu\mu} = \frac{2\pi^2 R_L^{s-2} v(1-\mu^2)|\mu|^{s-1}}{B_0^2}\sum_{j=\pm1} I_0^j \times\int_{-1}^1 \mathrm{d}\eta \frac{(1+\eta^2)|\eta-\frac{j\epsilon}{\mu}|^{s-1}}{[\eta^2+\Lambda(1-\eta^2)]^{(2+s)/2}} \times\sum_{n=1}^\infty n^{-s}\left(J_n'\left(n\frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu\eta-j\epsilon|}\right)\right)^2 \times\left[H\left[\frac{\mathrm{sgn}(\Omega)}{j\epsilon-\mu\eta}\right]+H\left[\frac{\mathrm{sgn}(\Omega)}{\mu\eta-j\epsilon}\right]\right] \quad (22)$$

together with

$$k_{\min} \le \frac{n}{R_L|\mu\eta-j\epsilon|} \le k_{\max} \quad (23)$$

where we introduced the cosmic ray particle gyroradius $R_L = v/|\Omega|$.

Likewise we obtain

$$D^G_{pp} = p^2\epsilon^2 D^G_{\mu\mu} \quad (24)$$

and

$$D^G_{\mu p} = \frac{4\pi^2 R_L^{s-2} p V_\mathrm{A}|\mu|^s}{B_0^2}\sum_{j=\pm1} jI_0^j \times\int_{-1}^1 \mathrm{d}\eta\frac{(1+\eta^2)|\eta-\frac{j\epsilon}{\mu}|^s}{[\eta^2+\Lambda(1-\eta^2)]^{(2+s)/2}}\left[\frac{\mu}{2}-\frac{\eta|\mu\eta-j\epsilon|}{1-\eta^2}\right] \times\sum_{n=1}^\infty n^{-s}J_n^2\left(n\frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu\eta-j\epsilon|}\right) \times\left[H\left[\frac{\mathrm{sgn}(\Omega)}{j\epsilon-\mu\eta}\right]+H\left[\frac{\mathrm{sgn}(\Omega)}{\mu\eta-j\epsilon}\right]\right] \quad (25)$$

again together with the restriction (23).

Without further information on the relative strengths of I_0^- to I_0^+ it is not possible to take the gyroresonance contributions much further. In Sect. 5 we treat with the symmetric case where $I_0^- = I_0^+$, to illuminate the changes in the Fokker-Planck coefficients brought about by the anisotropic nature of the plasma wave turbulence. However, for the transit-time contributions (Eqs. (19)–(21)) it is possible to evaluate the effects of anisotropy directly without needing to make any further assumptions on I_0^- and I_0^+. This aspect is discussed next.

4. Anisotropic transit-time effects

Introducing the ratio

$$A_\mathrm{TT}(\Lambda) \equiv D^T_{\mu\mu}(\Lambda)/D^T_{\mu\mu}(\Lambda=1) = D^T_{pp}(\Lambda)/D^T_{pp}(\Lambda=1) \quad (26)$$

we obtain from Eq. (20)

$$A_\mathrm{TT}(\Lambda) = \frac{[\frac{\epsilon^2}{\mu^2}+\Lambda(1-\frac{\epsilon^2}{\mu^2})]^{-(2+s)/2}}{J(\Lambda)} \quad (27)$$

in $\epsilon < |\mu|$.

Three cases provide insight into the anisotropic effects:

(i) weak anisotropy $\Lambda = 1 - r, |r| \ll 1$;
(ii) strongly ribbon-like anisotropy $\Lambda \gg 1$;
(iii) strongly perpendicular anisotropy $\Lambda \ll 1$.

Consider each in turn.

4.1. Weak anisotropy ($\Lambda = 1 - r, |r| \ll 1$)

Here

$$A_\mathrm{TT}(\Lambda) \simeq \left[1+r\left(1+\frac{s}{2}\right)\left(1-\frac{\epsilon^2}{\mu^2}\right)\right]/\left[1+r\frac{s+2}{3}\right] \simeq 1+r\left(1+\frac{s}{2}\right)\left(\frac{1}{3}-\frac{\epsilon^2}{\mu^2}\right) \quad (28)$$

in $\epsilon < |\mu|$ so that the anisotropy component changes sign as μ crosses $\pm\sqrt{3}\epsilon$.

4.2. Strongly ribbon-like anisotropy ($\Lambda \gg 1$)

According to Eq. (43) of Lerche & Schlickeiser (2001) the integral (5) for $\Lambda \gg 1$ is approximately $J(\Lambda \gg 1) \simeq (s\Lambda)^{-1}$. Moreover, in this case $\epsilon^2/\mu^2 \ll \Lambda(1-\epsilon^2/\mu^2)$ except from the small range of μ in $\epsilon \le |\mu| \le \epsilon(1+\frac{1}{2\Lambda})$. Then

$$A_\mathrm{TT}(\Lambda) \simeq \times\begin{cases} s\Lambda\left(\frac{|\mu|}{\epsilon}\right)^{2+s} & \text{for } \epsilon \le |\mu| \le \epsilon(1+\frac{1}{2\Lambda}) \\ s\Lambda^{-s/2}\left[1-\frac{\epsilon^2}{\mu^2}\right]^{-(2+s)/2} & \text{for } \epsilon(1+\frac{1}{2\Lambda}) < |\mu| \le 1 \end{cases}. \quad (29)$$

Thus, in most of the range of μ, $A_\mathrm{TT} = \mathrm{O}(\Lambda^{s/2}) \ll 1$ and only in a very narrow range $|\mu| \cong \epsilon$, is $A_\mathrm{TT} = \mathrm{O}(\Lambda) \gg 1$. Consequently, in $\epsilon(1+\frac{1}{2\Lambda}) < |\mu| \le 1$

$$D^T_{\mu\mu}(\Lambda) \simeq s\Lambda^{-s/2}D^T_{\mu\mu}(\Lambda=1) \ll D^T_{\mu\mu}(\Lambda=1). \quad (30)$$

Thus the pitch angle transit-time Fokker-Planck component is massively reduced as is D^T_{pp}, for most particles. Only the small fraction of particles occupying the range $\epsilon(1+\frac{1}{2\Lambda}) < |\mu| \le 1$ have an enhanced transit-time contribution.

4.3. Strongly perpendicular anisotropy ($\Lambda \ll 1$)

Correcting Eq. (30) of Lerche & Schlickeiser (2001) the integral (5) for $\Lambda \ll 1$ is approximately $J(\Lambda \ll 1) \simeq (s+2)\Lambda^{-(s+1)/2}/(s+1)$.

With respect to the first factor in Eq. (27), in this case $\Lambda \ll 1$ the factor ϵ^2/μ^2 dominates $\Lambda(1-\epsilon^2/\mu^2)$ except when $1 \geq |\mu| \geq \epsilon\Lambda^{-1/2}$. The outer pair of inequalities require $\Lambda > \epsilon^2$ to be obeyed. So two limits exist: either (α) $1 \gg \epsilon^2 \gg \Lambda$, in which case the factor ϵ^2/μ^2 dominates $\Lambda(1-\epsilon^2/\mu^2)$ everywhere; or (β) $1 \gg \Lambda \geq \epsilon^2$, in which case the factor ϵ^2/μ^2 dominates $\Lambda(1-\epsilon^2/\mu^2)$ in $\epsilon \leq |\mu| \leq \epsilon\Lambda^{-1/2} \leq 1$ while $\Lambda(1-\epsilon^2/\mu^2)$ dominates in $\epsilon\Lambda^{-1/2} \leq |\mu| \leq 1$. Consider each case in turn.

4.3.1. $\Lambda \ll \epsilon^2 \ll 1$

Here for all values of $|\mu| > \epsilon$

$$A_{\rm TT}(\Lambda) \simeq \frac{s+1}{s+2}\frac{|\mu|}{\epsilon}\left[\frac{\mu^2\Lambda}{\epsilon^2}\right]^{(1+s)/2} \ll 1 \tag{31}$$

which is much smaller unity.

4.3.2. $\epsilon^2 \leq \Lambda \ll 1$

In this case Eq. (31) holds in the range $\epsilon \leq |\mu| \leq \epsilon\Lambda^{-1/2}$ whereas in the range $\epsilon\Lambda^{-1/2} < |\mu| \leq 1$

$$A_{\rm TT}(\Lambda) \simeq \frac{s+1}{s+2}\Lambda^{-1/2}[1-\frac{\epsilon^2}{\mu^2}]^{-(2+s)/2} \simeq \Lambda^{-1/2}\frac{s+1}{s+2} \tag{32}$$

which is larger unity.

5. Anisotropic gyroresonance effects

Because the individual wave intensity components I_0^+ and I_0^- do not enter the expressions for $D^G_{\mu\mu}$ and $D^G_{\mu p}$ as the simple sum $I_0^+ + I_0^-$, occurring for the transit-time components, one has to assume further simplifications in order to evaluate the respective contributions to the Fokker-Planck coefficients. SM argued that the most important situation to evaluate was that of symmetric wave intensities when $I_0^+ = I_0^- = I_{\rm tot}/2$. We follow that prescription here so that the direct effects of the anisotropic power spectrum are illustrated relative to a conventional scenario.

5.1. Rate of adiabatic deceleration

From Eq. (25) we obtain

$$D^G_{\mu p} = \frac{4\pi^2 R_L^{s-2} p\epsilon v|\mu|^{s+1} I_{\rm tot}}{B_0^2}\sum_{n=1}^{\infty} n^{-s}$$
$$\times \int_{-1}^{1} {\rm d}\eta \frac{\eta(1+\eta^2)}{(1-\eta^2)[\eta^2+\Lambda(1-\eta^2)]^{(2+s)/2}}$$
$$\times \left(|\eta + \frac{\epsilon}{\mu}|^{s+1} J_n^2\left(n\frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu\eta+\epsilon|}\right) H_{{\rm min},-}H_{{\rm max},}\right.$$
$$\left. -|\eta - \frac{\epsilon}{\mu}|^{s+1} J_n^2\left(n\frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu\eta-\epsilon|}\right) H_{{\rm min},+}H_{{\rm max},+}\right) \tag{33}$$

$$H_{{\rm min},\pm} = H\left[\frac{n}{R_L|\mu\eta \mp \epsilon|} - k_{\rm min}\right] \tag{34}$$

and

$$H_{{\rm max},\pm} = H\left[k_{\rm max} - \frac{n}{R_L|\mu\eta \mp \epsilon|}\right]. \tag{35}$$

Note that $D^G_{\mu p}(-\mu) = -D^G_{\mu p}(\mu)$ is antisymmetric in μ, so that the rate of adiabatic acceleration D in Eq. (8)

$$D = \frac{3v}{4p}\int_{-1}^{1}{\rm d}\mu(1-\mu^2)\frac{D_{\mu p}}{D_{\mu\mu}} = \frac{3v}{4p}\int_{-1}^{1}{\rm d}\mu(1-\mu^2)\frac{D^G_{\mu p}}{D_{\mu\mu}} = 0 \tag{36}$$

is identically zero in the symmetric wave intensity case $I_0^+ = I_0^- = I_{\rm tot}/2$. In deriving this result we have used Eq. (16) that there is no transit-time contribution to $D_{\mu p}$.

Note that as $\epsilon \to 0$, the integral over η in Eq. (33) also tends to zero, so that $D^G_{\mu p}$ is small compared to $D^G_{\mu\mu}$.

5.2. Pitch-angle Fokker-Planck coefficient

From Eq. (22)

$$D^G_{\mu\mu} = \frac{2\pi^2 R_L^{s-2} v(1-\mu^2)|\mu|^{s-1} I_{\rm tot}}{B_0^2}\sum_{n=1}^{\infty} n^{-s}$$
$$\times \int_0^1 {\rm d}\eta \frac{(1+\eta^2)}{[\eta^2+\Lambda(1-\eta^2)]^{(2+s)/2}}$$
$$\times \left(|\eta - \frac{\epsilon}{\mu}|^{s-1}\left[J_n'\left(n\frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu\eta-\epsilon|}\right)\right]^2 H_{{\rm min},+}H_{{\rm max},+}\right.$$
$$\left. + |\eta + \frac{\epsilon}{\mu}|^{s-1}\left[J_n'\left(n\frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu\eta+\epsilon|}\right)\right]^2 H_{{\rm min},-}H_{{\rm max},-}\right). \tag{37}$$

We further reduce Eq. (37) by changing variables in the first term of the bracket to $w = (\eta\mu - \epsilon)/n$ with $-\epsilon/n \leq w \leq (\mu - \epsilon)/n$ while in the second term one writes $q = (\eta\mu + \epsilon)/n$ with $\epsilon/n \leq w \leq (\mu+\epsilon)/n$. We obtain

$$D^G_{\mu\mu} = \frac{2\pi^2 R_L^{s-2} v(1-\mu^2) I_{\rm tot}}{\mu B_0^2}$$
$$\times \sum_{n=1}^{\infty}\left[\int_{-\frac{\epsilon}{n}}^{(\mu-\epsilon)/n} {\rm d}q \frac{(1+\eta_+^2)}{[\eta_+^2+\Lambda(1-\eta_+^2)]^{(2+s)/2}}|q|^{s-1}\right.$$
$$\times \left[J_n'\left(\frac{(1-\mu^2)^{1/2}(1-\eta_+^2)^{1/2}}{|q|}\right)\right]^2$$
$$\times H\left[\frac{1}{R_L|q|} - k_{\rm min}\right] H\left[k_{\rm max} - \frac{1}{R_L|q|}\right]$$
$$+ \int_{\frac{\epsilon}{n}}^{(\mu+\epsilon)/n} {\rm d}q \frac{(1+\eta_-^2)}{[\eta_-^2+\Lambda(1-\eta_-^2)]^{(2+s)/2}}|q|^{s-1}$$
$$\times \left[J_n'\left(\frac{(1-\mu^2)^{1/2}(1-\eta_-^2)^{1/2}}{|q|}\right)\right]^2$$
$$\left.\times H\left[\frac{1}{R_L|q|} - k_{\rm min}\right] H\left[k_{\rm max} - \frac{1}{R_L|q|}\right]\right] \tag{38}$$

where $\eta_+(q)=(nq+\epsilon)/\mu$ and $\eta_-(q)=(nq-\epsilon)/\mu$. Eq. (38) is discussed in Appendix A.

5.2.1. Small values $|\mu|\le\epsilon$

As shown in Appendix A, for small pitch angle cosines $|\mu|\le\epsilon$ we obtain

$$D^G_{\mu\mu}(|\mu|\le\epsilon)\simeq\frac{(\delta B)^2}{B_0^2}A_{G1}(\Lambda)R_L^{s-2}v\epsilon^s\left[\int_{k_{\min}}^{k_{\max}}\mathrm{d}k\,k^{-s}\right]^{-1}$$
$$H[E-E_{\min}]H[E_{\max}-E]\sum_{n=1}^{\infty}n^{-(1+s)}[1-(-1)^n\sin(2n/\epsilon)] \qquad (39)$$

with

$$E_{\min}\equiv\frac{Zm_pc^2}{2\pi}\frac{l_{\min}\omega_{p,i}}{c};\qquad E_{\max}=\frac{l_{\max}}{l_{\max}}E_{\min} \qquad (40)$$

involving the inner and outer scale of the turbulence spectrum and the interstellar ion skin length, respectively. We introduced the gyroresonance anisotropy ratio for small pitch angle cosines

$$A_{G1}(\Lambda)=\left[\Lambda^{(2+s)/2}J(\Lambda)\right]^{-1} \qquad (41)$$

in terms of the integral (5). Obviously

$$D^G_{\mu\mu}(\Lambda,|\mu|\le\epsilon)=A_{G1}(\Lambda)D^G_{\mu\mu}(\Lambda=1,|\mu|\le\epsilon) \qquad (42)$$

the ratio A_{G1} relates the respective Fokker-Planck coefficients for general anisotropy ($\Lambda\ne1$) to the isotropic ($\Lambda=1$) one.

For isotropic turbulence ($\Lambda=1$) Eq. (39) (apart from the factor $3\pi/2$) agrees with Eqs. (44) and (58a) of SM.

5.2.2. Large values $|\mu|>\epsilon$

For large pitch angles $|\mu|>\epsilon$ we restrict our analysis to cosmic ray particles with gyroradii less than $R_L<l_{\max}/2\pi$. In this case from Appendix A we obtain from Eq. (108)

$$D^G_{\mu\mu}(\Lambda,\epsilon<|\mu|\le2^{-1/2})=A_{G2}(\Lambda)D^G_{\mu\mu}(\Lambda=1,\epsilon<|\mu|\le2^{-1/2}) \qquad (43)$$

and

$$D^G_{\mu\mu}(\Lambda,|\mu|>2^{-1/2})=A_{G3}(\Lambda)D^G_{\mu\mu}(\Lambda=1,|\mu|>2^{-1/2}) \qquad (44)$$

with

$$D^G_{\mu\mu}(\Lambda=1,\epsilon<|\mu|\le2^{-1/2})=\frac{\zeta(s+1)g(s)}{2}\frac{(\delta B)^2}{B_0^2}R_L^{s-2}v|\mu|^{s+1}\left[\int_{k_{\min}}^{k_{\max}}\mathrm{d}k\,k^{-s}\right]^{-1} \qquad (45)$$

where

$$g(s)=\pi^{1/2}\frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]}+2^{-s/2}\times\left[{}_2F_1\left(1+\frac{s}{2},1;\frac{5+s}{2};\frac{1}{2}\right)+{}_2F_1\left(1+\frac{s}{2},1;\frac{3}{2};\frac{1}{2}\right)\right] \qquad (46)$$

and

$$D^G_{\mu\mu}(\Lambda=1,|\mu|>2^{-1/2})=\frac{\pi}{4s}\frac{(\delta B)^2}{B_0^2}R_L^{s-2}v(1-\mu^2)|\mu|^{s-1}\left[\int_{k_{\min}}^{k_{\max}}\mathrm{d}k\,k^{-s}\right]^{-1}, \qquad (47)$$

respectively. The two gyroresonance anisotropy ratios are given by

$$A_{G2}(\Lambda)=[g(s)J(\Lambda)]^{-1}\left[\frac{\pi^{1/2}}{\Lambda^{1/2}}\frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]}+2[1+\Lambda]^{-(2+s)/2}\times\left[{}_2F_1\left(1+\frac{s}{2},1;\frac{5+s}{2};\frac{1}{1+\Lambda}\right)+{}_2F_1\left(1+\frac{s}{2},1;\frac{3}{2};\frac{\Lambda}{1+\Lambda}\right)\right]\right] \qquad (48)$$

and

$$A_{G3}(\Lambda)=[\Lambda J(\Lambda)]^{-1}. \qquad (49)$$

It is remarkable that the three gyroresonance anisotropy ratios (Eqs. (41), (48) and (49)) are independent of cosmic ray particle properties and solely determined by the turbulence parameters s and Λ.

With the asymptotic behaviour of $J(\Lambda)$ for small and large arguments we immediately find for the asymptotic behaviour of the two gyroresonance anisotropy factors

$$A_{G1}(\Lambda)\simeq\begin{cases}\frac{s+1}{s+2}\Lambda^{-1/2} & \text{for }\Lambda\ll1\\ s\Lambda^{-s/2} & \text{for }\Lambda\gg1\end{cases} \qquad (50)$$

$$A_{G3}(\Lambda)\simeq\begin{cases}\frac{s+1}{s+2}\Lambda^{(s-1)/2} & \text{for }\Lambda\ll1\\ s & \text{for }\Lambda\gg1\end{cases}. \qquad (51)$$

The asymptotics of the anisotropy factor A_{G2} is more involved. For $\Lambda\ll1$ the two hypergeometric functions in Eq. (48) can be approximated as

$${}_2F_1\left(1+\frac{s}{2},1;\frac{5+s}{2};\frac{1}{1+\Lambda}\right)\simeq{}_2F_1\left(1+\frac{s}{2},1;\frac{5+s}{2};1\right)=3+s$$

and

$${}_2F_1\left(1+\frac{s}{2},1;\frac{3}{2};\frac{\Lambda}{1+\Lambda}\right)\simeq{}_2F_1\left(1+\frac{s}{2},1;\frac{3}{2};0\right)=1$$

so that the second factor in the bracket of Eq. (48) can be neglected, leaving

$$A_{G2}(\Lambda\ll1)\simeq\frac{\pi^{1/2}}{g(s)}\frac{\Gamma[\frac{3+s}{2}]}{\Gamma[\frac{4+s}{2}]}\Lambda^{s/2}. \qquad (52)$$

In the opposite case $\Lambda \gg 1$ the first hypergeometric function in Eq. (48) is approximated as

$$ {}_2F_1\left(1+\frac{s}{2},1;\frac{5+s}{2};\frac{1}{1+\Lambda}\right)\simeq {}_2F_1\left(1+\frac{s}{2},1;\frac{5+s}{2};0\right)=1 $$

whereas the second is written as the integral

$$ {}_2F_1\left(1+\frac{s}{2},1;\frac{3}{2};z\right)=\frac{1}{2}\int_0^1 \mathrm{d}t(1-t)^{-1/2}(1-zt)^{-(2+s)/2} \quad (53) $$

with $z=1-\xi$ where $\xi=(1+\Lambda)^{-1}\ll 1$. Substituting $x=z(1-t)/(1-z)$ in Eq. (53) we derive

$$ {}_2F_1(1+\frac{s}{2},1;\frac{3}{2};1-\xi)=\frac{\xi^{-(1+s)/2}}{2(1-\xi)^{1/2}} \times\int_0^{(1-\xi)/\xi}\mathrm{d}x\,x^{-1/2}(1+x)^{-(2+s)/2}\simeq\frac{\xi^{-(1+s)/2}}{2(1-\xi)^{1/2}} \times\int_0^{\infty}\mathrm{d}x\,x^{-1/2}(1+x)^{-(2+s)/2}\simeq\frac{\pi^{1/2}}{2}\frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]}\xi^{-(1+s)/2}. \quad (54) $$

The replacement of the upper integration boundary with ∞ is allowed because $\xi \ll 1$. Accordingly, we obtain

$$ A_{G2}(\Lambda\gg 1)\simeq\frac{2s\pi^{1/2}}{g(s)}\frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]}\Lambda^{1/2}. \quad (55) $$

In summary then

$$ A_{G2}(\Lambda)\simeq\frac{\pi^{1/2}}{g(s)}\frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]}\begin{cases}\frac{s+1}{s+2}\Lambda^{s/2} & \text{for } \Lambda\ll 1\\ 2s\Lambda^{1/2} & \text{for } \Lambda\gg 1\end{cases} \quad (56) $$

complementing Eqs. (50) and (51).

We are now in the position to discuss the influence of the turbulence anisotropy Λ on the ratio of the contributions from transit-time damping and gyroresonances in the pitch-angle interval $|\mu|>\epsilon$.

6. Comparison of transit-time damping and gyroresonance contributions to particle scattering

Transit-time damping does not contribute to the scattering of particles in the interval $|\mu|<\epsilon$ where the scattering relies solely on the gyroresonant contribution (SM). Outside this interval we can calculate the ratio of the contributions from transit-time damping and gyroresonances as

$$ R_{2,3}(\Lambda)\equiv\frac{D^T_{\mu\mu}(\Lambda)}{D^G_{\mu\mu}(\Lambda)}=r_{2,3}(\mu)\frac{A_{\rm TT}(\Lambda)}{A_{G2,3}(\Lambda)} \quad (57) $$

where the indices $2,3$ refer to the intervals $\epsilon\le|\mu|\le 2^{-1/2}$ and $|\mu|>2^{-1/2}$, respectively. The functions

$$ r_{2,3}(\Lambda)\equiv\frac{D^T_{\mu\mu}(\Lambda=1)}{D^G_{\mu\mu}(\Lambda=1)} \quad (58) $$

refer to the corresponding ratios for isotropic turbulence, and are obtained from Eqs. (19), (45) and (47) under the assumptions that have been made as

$$ r_2(\mu)\simeq O_2\left[1-\frac{\epsilon^2}{\mu^2}\right]^{s/2}\mu^{-(2+s)}; \qquad r_3(\mu)\simeq O_3[1-\mu^2]^{s/2}\mu^{-s} \quad (59) $$

where O_2 and O_3 are factors of order unity. Whereas $r_3(\mu)$ is of order O_3 in the whole range $|\mu|>2^{-1/2}$, the function $r_2(\mu)$ attains its maximum value

$$ r_{2,\max}=O_1\epsilon^{-(2+s)}\gg 1 \quad (60) $$

at $|\mu_0|=\epsilon[2(s+1)/(s+2)]^{1/2}$ where

$$ O_1=\frac{O_2(s+2)}{2(s+1)}[\frac{s(s+2)}{4(s+1)^2}]^{s/2} $$

is again of order unity. Eq. (60) reproduces the result of SM that for isotropic turbulence transit-time damping provides the dominant contribution to pitch angle scattering in the interval $|\mu|>\epsilon$.

6.1. Interval $\epsilon<|\mu|\le 2^{-1/2}$

Approximating in this interval the variation of the function $r_2(\mu)$ by its maximum value (60) we obtain for the ratio of the contributions from transit-time damping and gyroresonances

$$ R_2(\Lambda)\simeq O_1\epsilon^{-(2+s)}\frac{A_{\rm TT}(\Lambda)}{A_{G2}(\Lambda)} \quad (61) $$

which can be reduced further using the approximations (29), (31), (32) and (56).

6.1.1. Strongly perpendicular anisotropy ($\Lambda\ll 1$)

For strongly perpendicular anisotropy we obtain

$$ R_2(\Lambda\ll 1)\simeq\frac{O_1 g(s)\Gamma[\frac{2+s}{2}]}{\pi^{1/2}\Gamma[\frac{2+s}{2}]}\times\begin{cases}\mu^{2+s}\epsilon^{-2(2+s)}\Lambda^{1/2} & \text{for } \Lambda\ll\epsilon^2\ll 1\\ \frac{s+1}{2s(s+2)}\epsilon^{-(2+s)}\Lambda^{-1/2} & \text{for } \epsilon^2\le\Lambda\ll 1\end{cases} \quad (62) $$

which is much larger unity unless $\Lambda\le\epsilon^{2(2+s)}$ is extremely small.

6.1.2. Strongly ribbon-like anisotropy ($\Lambda\gg 1$)

For strongly parallel anisotropy we derive

$$ R_2(\Lambda\gg 1)\simeq\frac{O_1 g(s)\Gamma[\frac{2+s}{2}]}{2\pi^{1/2}\Gamma[\frac{2+s}{2}]}\times\begin{cases}\epsilon^{-(2+s)}\Lambda^{1/2} & \text{for } \epsilon\le|\mu|\le\epsilon(1+\frac{1}{2\Lambda})\\ \epsilon^{-(2+s)}\Lambda^{-(s+1)/2} & \text{for } |\mu|>\epsilon(1+\frac{1}{2\Lambda})\end{cases} \quad (63) $$

which in the small interval $\epsilon\le|\mu|\le\epsilon(1+\frac{1}{2\Lambda})$ is always much larger unity. Outside this interval, i.e. $|\mu|>\epsilon(1+\frac{1}{2\Lambda})$ the ratio $R_2(\Lambda\gg 1)$ is much larger unity unless $\Lambda>\epsilon^{-2(2+s)/(s+1)}$ becomes extremely large.

6.2. Interval $|\mu| > 2^{-1/2}$

In this interval we approximate the variation of the function $r_3(\mu)$ by its maximum value O_3 so that we obtain for the ratio of the contributions from transit-time damping and gyroresonances

$$R_3(\Lambda) \simeq O_3 \frac{A_{\rm TT}(\Lambda)}{A_{G3}(\Lambda)}. \tag{64}$$

6.2.1. Strongly perpendicular anisotropy ($\Lambda \ll 1$)

For strongly perpendicular anisotropy we obtain

$$R_3(\Lambda \ll 1) \simeq \frac{O_3 g(s)\Gamma[\frac{2+s}{2}]}{\pi^{1/2}\Gamma[\frac{2+s}{2}]} \times \begin{cases} \mu^{2+s}\epsilon^{2+s}\Lambda^{1/2} & \text{for } \Lambda \ll \epsilon^2 \ll 1 \\ \frac{s+1}{2s(s+2)}\Lambda^{-1/2} & \text{for } \epsilon^2 \le \Lambda \ll 1 \end{cases} \tag{65}$$

which is much larger unity unless $\Lambda \le \epsilon^{2(2+s)}$ is extremely small.

6.2.2. Strongly ribbon-like anisotropy ($\Lambda \gg 1$)

For strongly parallel anisotropy we derive

$$R_3(\Lambda \gg 1) \simeq \frac{O_3 g(s)\Gamma[\frac{2+s}{2}]}{2\pi^{1/2}\Gamma[\frac{2+s}{2}]}\Lambda^{-(s+1)/2} \tag{66}$$

which is much smaller unity.

6.3. Interlude

Summarizing our results in short:
(a) For massively parallel ($\Lambda \gg 1$) situations, the ratio of the transit-time contribution to the gyroresonance contribution to pitch-angle scattering in the interval $|\mu| > \epsilon$ of cosmic ray particles with gyroradii $R_L < l_{\max}/2\pi$ behaves as follows:
– for large $|\mu| > 2^{-1/2}$ the ratio is smaller than unity indicating that the gyroresonance contribution dominates the transit-time damping contribution,
– in the small interval $\epsilon \le |\mu| \le \epsilon(1+\frac{1}{2\Lambda})$ the ratio is larger than unity indicating that the transit-time contribution dominates the gyroresonance contribution,
– in the interval $\epsilon(1+\frac{1}{2\Lambda}) < |\mu| \le 2^{-1/2}$ the ratio is larger than unity (i.e. dominance of the transit-time damping contribution) for anisotropy values smaller than $1 \ll \Lambda \le \Lambda_l \equiv \epsilon^{-2(2+s)/(s+1)}$ whereas for extremely large values of $\Lambda > \Lambda_l$ the ratio is smaller than unity (i.e. dominance of the gyroresonance contribution).
(b) For massively perpendicular ($\Lambda \ll 1$) situations, the ratio of the transit-time contribution to the gyroresoance contribution to pitch-angle scattering in the interval $|\mu| > \epsilon$ of cosmic ray particles with gyroradii $R_L < l_{\max}/2\pi$ is much larger than unity for anisotropy values larger than $\epsilon^{2(2+s)} \equiv \Lambda_s \le \Lambda \ll 1$, indicating that the transit-time damping contribution dominates the gyroresonance contribution.
For extremely small anisotropy values $\Lambda < \Lambda_s \ll 1$ the ratio is smaller than unity indicating the dominance of the gyroresonance contribution over the transit-time damping contribution.

6.4. Cosmic ray scattering in the interstellar medium

Using the estimates of the Alfven speed in the diffuse interstellar medium of $V_{\rm A} \simeq 3 \times 10^6$ cm s^{-1} (Minter & Spangler 1997) yields the value $\epsilon = V_{\rm A}/v \simeq V_{\rm A}/c = 10^{-4}$ for relativistic cosmic ray particles. With a turbulence spectral index of $s = 5/3$ (Rickett 1990) we obtain for $\Lambda_l = \epsilon^{-2(2+s)/(s+1)} = \epsilon^{-11/4} = 10^{11}$ and $\Lambda_s = \epsilon^{2(2+s)} = \epsilon^{22/3} = 10^{-88/3} = 2 \times 10^{-29}$, respectively.

Now, estimates of the anisotropy parameter Λ in the strongly parallel situation ($\Lambda \gg 1$) based on linear Landau damping balancing radiative loss in the diffuse interstellar medium, provide the value $\Lambda \simeq 7400$ (Lerche & Schlickeiser 2001) which is much smaller than Λ_l. Hence, it would seem that in the diffuse interstellar medium the transit-time damping contribution to $D_{\mu\mu}$ is dominant in the pitch-angle angle interval $\epsilon \le |\mu| \le 2^{-1/2}$ whereas the gyroresonant contribution dominates in the interval $\mu| > 2^{-1/2}$. The same conclusion holds in HII-regions (the fluctiferous domain of Spangler 1991), for which Lerche & Schlickeiser (2001) estimated $\Lambda \simeq 17.7$.

Estimates of the anisotropy parameter Λ in the strongly perpendicular situation ($\Lambda \ll 1$) based on linear Landau damping balancing radiative loss in the diffuse interstellar medium, provide the value $\Lambda \simeq 10^{-6}$ (Lerche & Schlickeiser 2001) which is much larger than Λ_s. The transit-time damping contribution then dominates the gyroresonance contribution throughout the whole pitch-angle interval $|\mu| \ge \epsilon$ in the diffuse interstellar medium. The same conclusion holds in HII-regions, for which Lerche & Schlickeiser (2001) estimated $\Lambda \simeq 10^{-3}$ in this case.

These estimates have direct consequences for the cosmic ray transport parameters in the interstellar medium, as the parallel mean free path and the momentum diffusion coeffient. However, before the parallel mean free path can be calculated, we have to determine the influence of the anisotropy parameter on the Fokker-Planck coefficients in case of shear Alfven waves, because interstellar plasma turbulence is a mixture of fast magnetosonic waves and shear Alfven waves (SM). This analysis will be the subject of the second paper of this series. Here, we restrict our analysis to the momentum diffusion coefficient which, for the relevant range $\Lambda_s \ll \Lambda \ll \Lambda_l$, is solely determined by the transit-time damping contribution.

7. Cosmic ray momentum diffusion from fast-mode waves

Using Eqs. (26), (19), (21) and (9) we obtain for the momentum diffusion coefficient of cosmic rays with gyroradii much less than $R_L \ll l_{\max}/2\pi$

$$A = \frac{\pi}{2}(s-1)c_1(s)\frac{(\delta B)^2}{B_0^2}(k_{\min}R_L)^{s-1}\frac{v\epsilon^2 p^2}{R_L}h(\Lambda,\epsilon,s) \tag{67}$$

with

$$c_1(s) = \int_0^\infty \mathrm{d}u\, u^{-(1+s)} J_1^2(u) = \frac{2^{1-s}s}{4-s^2}\frac{\Gamma[s]\Gamma[2-\frac{s}{2}]}{\Gamma^3[1+\frac{s}{2}]}$$

and the anisotropy function

$$h(\Lambda,\epsilon,s) \equiv \int_\epsilon^1 \mathrm{d}\mu\, A_{\mathrm{TT}}(\mu,\Lambda)\frac{1-\mu^2}{\mu}\left[1+\frac{\epsilon^2}{\mu^2}\right] \times \left[(1-\mu^2)\left(1-\frac{\epsilon^2}{\mu^2}\right)\right]^{s/2}. \quad (68)$$

We calculate the anisotropy function Eq. (68) using the respective approximations of the ratio A_{TT} from Sect. 4 in the relevant anisotropy range $\epsilon^2 \le \Lambda \le \Lambda_l$.

7.1. Isotropic turbulence $\Lambda = 1$

This case $A_{\mathrm{TT}} = 1$ has been considered before by SM who derived

$$h(\Lambda = 1) \simeq \ln \epsilon^{-1}. \quad (69)$$

7.2. Strongly parallel turbulence $1 \ll \Lambda \ll \Lambda_l$

Here we use Eq. (29) to obtain

$$h(1 \ll \Lambda \ll \Lambda_l, \epsilon, s) \simeq s\Lambda\epsilon^{-(2+s)}M_1 + \frac{s}{2}\Lambda^{-s/2}M_2 \quad (70)$$

with the two integrals

$$M_1 = \int_\epsilon^{\epsilon(1+\frac{1}{2\Lambda})} \mathrm{d}\mu\, \mu^{1+s}(1-\mu^2)\left[1+\frac{\epsilon^2}{\mu^2}\right] \times \left[(1-\mu^2)\left(1-\frac{\epsilon^2}{\mu^2}\right)\right]^{s/2} \quad (71)$$

and

$$M_2 = \int_{\epsilon(1+\frac{1}{2\Lambda})}^1 \mathrm{d}\mu\, \frac{1}{\mu}(1-\mu^2)^{(2+s)/2}\frac{1+\frac{\epsilon^2}{\mu^2}}{1-\frac{\epsilon^2}{\mu^2}} = \frac{1}{2}\int_{\epsilon^2(1+\frac{1}{\Lambda})}^1 \mathrm{d}z\, \frac{1}{z}\frac{z+\epsilon^2}{z-\epsilon^2}(1-z)^{(2+s)/2} \quad (72)$$

with the substitution $z = \mu^2$.

We note that the μ-integration interval in Eq. (71) is very small so that we approximate the integrand by its value at $\mu = \epsilon(1+\frac{1}{2\Lambda})$ to obtain approximately

$$M_1 \simeq \epsilon^{2+s}\Lambda^{-(2+s)/2}. \quad (73)$$

For the integral (72) we substitute $z = (1-\epsilon^2)t + \epsilon^2$ to derive

$$M_2 = \frac{1}{2}(1-\epsilon^2)^{(2+s)/2}\left[2\int_{\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}}^1 \mathrm{d}t\, t^{-1}(1-t)^{(2+s)/2} - \int_{\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}}^1 \mathrm{d}t\, \left[t+\frac{\epsilon^2}{1-\epsilon^2}\right]^{-1}(1-t)^{(2+s)/2}\right] \quad (74)$$

which after the substitution $t = 1-[1-\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}]y$ can be expressed in terms of hypergeometric functions,

$$M_2 = \frac{2}{4+s}(1-\epsilon^2)^{(2+s)/2}\left[1-\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}\right]^{(4+s)/2} \times \left[{}_2F_1\left(1,2+\frac{s}{2};3+\frac{s}{2};1-\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}\right) - \frac{(1-\epsilon^2)}{2} \times {}_2F_1\left(1,2+\frac{s}{2};3+\frac{s}{2};1-\epsilon^2\left(1+\frac{1}{\Lambda}\right)\right)\right]. \quad (75)$$

According to Eq. (15.3.10) of Abramowitz & Stegun (1972) we use

$${}_2F_1(1,2+\frac{s}{2};3+\frac{s}{2};z) = \frac{4+s}{2}\sum_{n=0}^\infty \frac{\Gamma[n+2+\frac{s}{2}]}{\Gamma[2+\frac{s}{2}]n!} \times \left(\psi(n+1)-\psi(n+2+\frac{s}{2})+\ln(1-z)\right)(1-z)^n \quad (76)$$

to approximate the two hypergeometric functions in Eq. (75) as

$${}_2F_1\left(1,2+\frac{s}{2};3+\frac{s}{2};1-\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}\right) \simeq -\frac{4+s}{2}\ln\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}.$$

and

$${}_2F_1\left(1,2+\frac{s}{2};3+\frac{s}{2};1-\epsilon^2\left(1+\frac{1}{\Lambda}\right)\right) \simeq -\frac{4+s}{2}\ln\epsilon^2\left(1+\frac{1}{\Lambda}\right).$$

We then find

$$M_2 \simeq (1-\epsilon^2)^{(2+s)/2}\left[1-\frac{\epsilon^2}{\Lambda(1-\epsilon^2)}\right]^{(4+s)/2} \times \left[-\ln\frac{\epsilon^2}{\Lambda(1-\epsilon^2)} + \frac{(1-\epsilon^2)}{2}\ln\epsilon^2\left(1+\frac{1}{\Lambda}\right)\right] \simeq \ln\epsilon^{-1} + \frac{1}{2\Lambda}. \quad (77)$$

Collecting terms in Eq. (70) we find

$$h(1 \ll \Lambda \ll \Lambda_l, \epsilon, s) \simeq s\Lambda^{-s/2}[1+\frac{1}{2}\ln\epsilon^{-1}] \simeq \frac{s}{2}\Lambda^{-s/2}\ln\epsilon^{-1} \quad (78)$$

which is strongly reduced compared to the isotropic value.

7.3. Strongly perpendicular turbulence $\epsilon^2 \ll \Lambda \ll 1$

Here we use Eqs. (31) and (32) to obtain

$$h(\epsilon^2 \ll \Lambda \ll 1, \epsilon, s) = \frac{s+1}{s+2}\Lambda^{-1/2} \times \left[(\epsilon\Lambda^{-1/2})^{-(2+s)}\int_\epsilon^{\epsilon\Lambda^{-1/2}} \mathrm{d}\mu\, \mu^{1+s}(1-\mu^2) \times \left[1+\frac{\epsilon^2}{\mu^2}\right]\left[(1-\mu^2)\left(1-\frac{\epsilon^2}{\mu^2}\right)\right]^{s/2} + \int_{\epsilon\Lambda^{-1/2}}^1 \mathrm{d}\mu \frac{(1-\mu^2)^{(2+s)/2}}{\mu}\frac{1+\frac{\epsilon^2}{\mu^2}}{1-\frac{\epsilon^2}{\mu^2}}\right]. \quad (79)$$

Substituting in the first integral $\mu = \epsilon x^{1/2}$ yields

$$h(\epsilon^2 \ll \Lambda \ll 1, \epsilon, s) = \frac{s+1}{s+2}\Lambda^{-1/2}\left[\frac{\Lambda^{(2+s)/2}}{2}M_3 + M_4\right] \quad (80)$$

where

$$M_3 = \int_1^{\Lambda^{-1}} \mathrm{d}x\,(x-1)^{s/2}\left(1+\frac{1}{x}\right)(1-\epsilon^2 x)^{(2+s)/2} \quad (81)$$

and

$$M_4 = \int_{\epsilon\Lambda^{-1/2}}^{1} \mathrm{d}\mu \frac{(1-\mu^2)^{(2+s)/2}}{\mu}\,\frac{1+\frac{\epsilon^2}{\mu^2}}{1-\frac{\epsilon^2}{\mu^2}}. \quad (82)$$

Because $\epsilon^2 \ll \Lambda$ the integral (81) is well approximated by

$$M_3 \simeq \int_1^{\Lambda^{-1}} \mathrm{d}x\,(x-1)^{s/2}\left(1+\frac{1}{x}\right) \leq 2 \times \int_1^{\Lambda^{-1}} \mathrm{d}x\,(x-1)^{s/2} = \frac{4}{2+s}(\Lambda^{-1}-1)^{(2+s)/2}. \quad (83)$$

Apart from the minor difference in the lower integration boundary the integral (82) is identical to the integral (72). According to Eq. (75) we obtain

$$M_4 = \frac{2}{4+s}(1-\epsilon^2)^{(2+s)/2}\left[1-\frac{\epsilon^2(\Lambda^{-1}-1)}{(1-\epsilon^2)}\right]^{(4+s)/2} \times \left[{}_2F_1\left(1, 2+\frac{s}{2}; 3+\frac{s}{2}; 1-\frac{\epsilon^2(\Lambda^{-1}-1)}{(1-\epsilon^2)}\right) - \frac{(1-\epsilon^2)}{2}\,{}_2F_1\left(1, 2+\frac{s}{2}; 3+\frac{s}{2}; 1-\frac{\epsilon^2}{\Lambda}\right)\right]\cdot \quad (84)$$

Using again Eq. (76) we find

$$M_4 \simeq (1-\epsilon^2)^{(2+s)/2}\left[1-\frac{\epsilon^2(\Lambda^{-1}-1)}{(1-\epsilon^2)}\right]^{(4+s)/2} \times \left[-\ln\frac{\epsilon^2(\Lambda^{-1}-1)}{(1-\epsilon^2)} + \frac{(1-\epsilon^2)}{2}\ln\frac{\epsilon^2}{\Lambda}\right] \simeq \frac{1}{2}\ln\frac{\Lambda}{\epsilon^2}\,\cdot \quad (85)$$

Collecting terms in Eq. (80) we obtain

$$h(\epsilon^2 \ll \Lambda \ll 1, \epsilon, s) = \frac{s+1}{s+2}\Lambda^{-1/2} \times \left[\frac{2}{2+s}(1-\Lambda)^{(2+s)/2} + \frac{1}{2}\ln\frac{\Lambda}{\epsilon^2}\right] \simeq \frac{s+1}{(s+2)}\Lambda^{-1/2}[\ln\epsilon^{-1} + \ln\Lambda^{1/2}] \quad (86)$$

which is enhanced compared to the isotropic value.

We summarize the asymptotic behaviour of the anisotropy function $h(\Lambda, \epsilon, s)$ in Table 1.

8. Summary and conclusions

Observations of interstellar scintillations, general theoretical considerations and comparison of interstellar radiative cooling in HII-regions and in the diffuse interstellar medium with linear Landau damping estimates for fast-mode decay, all strongly imply that the power spectrum of fast-mode wave turbulence in the interstellar medium must be highly anisotropic. It is not clear from the observations whether the turbulence spectrum is oriented mainly parallel or mainly perpendicular to the ambient magnetic field, either will satisfy the needs of balancing wave damping energy input against radiative cooling. This anisotropy must be included when transport of high energy cosmic rays in the Galaxy is discussed.

Table 1. Anisotropy function h for different anisotropy parameters.

Anisotropy parameter Λ	$h(\Lambda, \epsilon, s)$
$\Lambda = 1$	$\ln\epsilon^{-1}$
$\epsilon^2 \ll \Lambda \ll 1$	$\frac{s+1}{s+2}\Lambda^{-1/2}[\ln\epsilon^{-1} + \ln\Lambda^{1/2}]$
$1 \ll \Lambda \ll \epsilon^{-2(2+s)/(s+1)}$	$\frac{s}{2}\Lambda^{-s/2}\ln\epsilon^{-1}$

With this first paper we have started to evaluate the relevant cosmic ray transport parameters in the presence of anisotropic fast magnetosonic plasma wave turbulence. All technical details of the calculation of Fokker-Planck coefficients in this case are presented, in particular the deviations from the case of isotropic turbulence are identified. Using the estimates of the anisotropy parameter in the strongly parallel and perpendicular regimes, based on linear Landau damping balancing radiative loss in the diffuse interstellar medium, we have calculated the Fokker-Planck coefficients needed to infer the parallel mean free path, the rate of adiabatic deceleration and the momentum diffusion coefficient of cosmic ray particles. We show that in nearly all situations the pitch-angle scattering of relativistic cosmic rays by fast magnetosonic waves at pitch-angle cosines $|\mu| \geq V_A/c$ is dominated by the transit-time damping interaction.

These results have direct consequences for the cosmic ray transport parameters in the interstellar medium, as the parallel mean free path and the momentum diffusion coefficient. In order to calculate the parallel mean free path, we have to determine the influence of the anisotropy parameter on the Fokker-Planck coefficients in case of shear Alfven waves, because interstellar plasma turbulence is a mixture of fast magnetosonic waves and shear Alfven waves. This analysis will be the subject of the second paper of this series.

Without considering the influence of the anisotropy parameter on the Fokker-Planck coefficients in case of shear Alfven waves, we are able to calculate the momentum diffusion coefficient a_2 of cosmic ray particles by averaging the respective Fokker-Planck coefficient over the particle pitch-angle for the relevant anisotropy parameters within values of $10^{-8} \leq \Lambda \leq 10^{11}$. For strongly perpendicular turbulence ($\Lambda \ll 1$) the cosmic ray momentum diffusion coefficient is enhanced with respect to isotropic ($\Lambda = 1$) turbulence by the large factor $\simeq \Lambda^{-1/2}$. For strongly parallel turbulence ($\Lambda \gg 1$) the momentum diffusion coefficient is reduced with respect to isotropic turbulence by the large factor $2\Lambda^{s/2}/s$. This implies that the acceleration time scale of cosmic ray particles by momentum diffusion for anisotropic turbulence is shorter (strongly perpendicular turbulence) or longer (strongly parallel turbulence) by the same factors with respect to the case of isotropic turbulence. Hence, depending on small or large enough anisotropy factors Λ, reacceleration effects in the transport of galactic cosmic rays become much stronger ($\Lambda \ll 1$) or weaker ($\Lambda \gg 1$), respectively.

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9. Appendix A: Analysis of the gyroresonant pitch-angle Fokker-Planck coefficient (38)

9.1. Small values $|\mu| \le \epsilon$

For small pitch angles $|\mu| \le \epsilon$ we note that the q-integration intervals in Eq. (38) are very small so that we approximate the two integrands by their values at $q = -\epsilon/n$ and $q = \epsilon/n$, respectively, implying $\eta_+ = \eta_- = 1$. Equation (38) then reduces to

$$D^G_{\mu\mu}(|\mu| \le \epsilon) \simeq \frac{4\pi^2 R_L^{s-2} v \epsilon^{s-1} I_{\rm tot}}{B_0^2 \Lambda^{(2+s)/2}} \times \sum_{n=1}^{\infty} n^{-s} \left[J_n'(\frac{n}{\epsilon})\right]^2 H\left[\frac{1}{R_L\epsilon} - k_{\rm min}\right] H\left[k_{\rm max} - \frac{1}{R_L\epsilon}\right]. \quad (87)$$

For fast cosmic ray particles $\epsilon = V_{\rm A}/v \ll 1$ so that we may use the approximation of Bessel functions for large arguments (Abramowitz & Stegun 1972)

$$J_n(nz, z>1) \simeq \sqrt{\frac{2}{\pi n z}} \cos\left[nz - \frac{(2n+1)\pi}{4}\right] \quad (88)$$

implying

$$\begin{aligned}[J_n'(nz, z>1)]^2 &\simeq \frac{2}{\pi n z} \sin^2\left[nz - \frac{(2n+1)\pi}{4}\right] \\ &= \frac{1}{\pi n z}\left[1 - \cos\left[2nz - \frac{(2n+1)\pi}{2}\right]\right] \\ &= \frac{1}{\pi n z}\left[1 - \sin(2nz)\sin\left(\frac{(2n+1)\pi}{2}\right)\right] \\ &= \frac{1}{\pi n z}[1 - (-1)^n \sin(2nz)] \quad (89)\end{aligned}$$

for the argument $z = 1/\epsilon \gg 1$. We then obtain

$$D^G_{\mu\mu}(|\mu| \le \epsilon) \simeq \frac{4\pi R_L^{s-2} v \epsilon^{s} I_{\rm tot}}{B_0^2 \Lambda^{(2+s)/2}} \times \sum_{n=1}^{\infty} n^{-(1+s)} [1 - (-1)^n \sin(2n/\epsilon)] \times H\left[\frac{1}{R_L\epsilon} - k_{\rm min}\right] H\left[k_{\rm max} - \frac{1}{R_L\epsilon}\right]. \quad (90)$$

With Eqs. (3)–(4) for $I_{\rm tot}$ we find

$$D^G_{\mu\mu}(|\mu| \le \epsilon) \simeq \frac{(\delta B)^2}{B_0^2} A_{G1}(\Lambda) R_L^{s-2} v \epsilon^s \left[\int_{k_{\rm min}}^{k_{\rm max}} {\rm d}k\, k^{-s}\right]^{-1} \times \sum_{n=1}^{\infty} n^{-(1+s)} [1 - (-1)^n \sin(2n/\epsilon)] \times H\left[\frac{1}{R_L\epsilon} - k_{\rm min}\right] H\left[k_{\rm max} - \frac{1}{R_L\epsilon}\right] \quad (91)$$

where we introduced the gyroresonance anisotropy ratio for small pitch angle cosines

$$A_{G1}(\Lambda) = \left[\Lambda^{(2+s)/2} J(\Lambda)\right]^{-1} \quad (92)$$

in terms of the integral (5). Finally,we note that

$$\frac{1}{R_L\epsilon} = \frac{|\Omega_0|}{\gamma V_{\rm A}} = \frac{Z m_p}{\gamma m} \omega_{p,i}/c \quad (93)$$

can be expressed in terms of the ion skin length $c/\omega_{p,i}$. The two Heaviside step functions in Eq. (91) then imply the restriction on the value of the cosmic ray particle energy

$$E_{\rm min} \le E \le E_{\rm max} \quad (94)$$

with

$$E_{\rm min} \equiv \frac{Z m_p c^2}{2\pi} \frac{l_{\rm min}\omega_{p,i}}{c}; \qquad E_{\rm max} = \frac{l_{\rm max}}{l_{\rm max}} E_{\rm min} \quad (95)$$

involving the inner and outer scale of the turbulence spectrum and the interstellar ion skin length, respectively. Equation (91) then takes the form

$$\begin{aligned}D^G_{\mu\mu}(|\mu| \le \epsilon) \;\simeq\; &\frac{(\delta B)^2}{B_0^2} A_{G1}(\Lambda) R_L^{s-2} v \epsilon^s \left[\int_{k_{\rm min}}^{k_{\rm max}} {\rm d}k\, k^{-s}\right]^{-1} \\ &\times H[E - E_{\rm min}] H[E_{\rm max} - E] \\ &\times \sum_{n=1}^{\infty} n^{-(1+s)} [1 - (-1)^n \sin(2n/\epsilon)].\end{aligned} \quad (96)$$

9.2. Large values $|\mu| > \epsilon$

For large pitch angles $|\mu| > \epsilon$ we use again the fact that for fast cosmic ray particles $\epsilon = V_{\rm A}/v \ll 1$. Thus the lowest order approximation in powers of ϵ to Eq. (38) is

$$\begin{aligned}D^G_{\mu\mu} \simeq &\frac{4\pi^2 R_L^{s-2} v (1-\mu^2) I_{\rm tot}}{\mu B_0^2} \\ &\times \sum_{n=1}^{\infty} \int_0^{\mu/n} {\rm d}q \frac{(1+\eta_0^2)}{[\eta_0^2 + \Lambda(1-\eta_0^2)]^{(2+s)/2}} |q|^{s-1} \\ &\times \left[J_n'\left(\frac{(1-\mu^2)^{1/2}(1-\eta_0^2)^{1/2}}{|q|}\right)\right]^2 \\ &\times H\left[\frac{1}{R_L|q|} - k_{\rm min}\right] H\left[k_{\rm max} - \frac{1}{R_L|q|}\right]\end{aligned} \quad (97)$$

where $\eta_0 = nq/\mu$. We first note the symmetry in Eq. (97).

$$D^G_{\mu\mu}(-\mu) = D^G_{\mu\mu}(\mu) \quad (98)$$

so that we can restrict our discussion on positive values of μ.

Next, the two step functions tell us that the integration range of q is limited to

$$U \leq q \leq \frac{l_{\max}}{l_{\min}} U; \quad \text{with } U = (R_L k_{\max})^{-1} = \frac{l_{\min}}{2\pi R_L} \cdot \quad (99)$$

Comparing these boundaries with the upper integration limit μ/n in Eq. (97) restricts the possible values of n and yields

$$D^G_{\mu\mu} \simeq \frac{4\pi^2 R_L^{s-2} v(1-\mu^2) I_{\rm tot}}{\mu B_0^2} \times \left(\sum_{n=1}^{\frac{l_{\min}\mu}{l_{\max}U}} \int_U^{\frac{l_{\max}U}{l_{\min}}} {\rm d}q \frac{(1+\eta_0^2)}{[\eta_0^2 + \Lambda(1-\eta_0^2)]^{(2+s)/2}} q^{s-1} \times \left[J_n' \left(\frac{(1-\mu^2)^{1/2}(1-\eta_0^2)^{1/2}}{|q|} \right) \right]^2 + \sum_{n=\frac{l_{\min}\mu}{l_{\max}U}}^{\mu/U} \int_U^{\mu/n} {\rm d}q \frac{(1+\eta_0^2)}{[\eta_0^2 + \Lambda(1-\eta_0^2)]^{(2+s)/2}} q^{s-1} \times \left[J_n' \left(\frac{(1-\mu^2)^{1/2}(1-\eta_0^2)^{1/2}}{|q|} \right) \right]^2 \right) \cdot \quad (100)$$

Changing from the integration variable q to $\eta = nq/\mu$ we derive

$$D^G_{\mu\mu} \simeq \frac{2\pi^2 R_L^{s-2} v(1-\mu^2)|\mu|^{s-1} I_{\rm tot}}{B_0^2} W(\Lambda, \mu, s, n, U) \quad (101)$$

with the integral

$$W(\Lambda, \mu, s, n, U) \equiv 2 \left(\sum_{n=1}^{\frac{l_{\min}\mu}{l_{\max}U}} n^{-s} \int_{nU/\mu}^{\frac{n l_{\max} U}{\mu l_{\min}}} {\rm d}\eta f_1(\eta) + \sum_{n=\frac{l_{\min}\mu}{l_{\max}U}}^{\mu/U} n^{-s} \int_{nU/\mu}^{1} {\rm d}\eta f_1(\eta) \right) \quad (102)$$

where

$$f_1(\eta) = \frac{(1+\eta^2)}{[\eta^2 + \Lambda(1-\eta^2)]^{(2+s)/2}} |\eta|^{s-1} \times \left[J_n' \left(n \frac{(1-\mu^2)^{1/2}(1-\eta^2)^{1/2}}{|\mu|\eta} \right) \right]^2 \cdot \quad (103)$$

Now it is convenient to introduce the tangent of the wave propagation angle $T = \sqrt{1-\eta^2}/\eta$ and the absolute value of the tangent of the pitch angle $M = \sqrt{1-\mu^2}/|\mu|$, respectively. With $x = T^2$ the integral (102) becomes

$$W(\Lambda, M, s, U) = \sum_{n=1}^{\frac{l_{\min}\mu}{l_{\max}U}} n^{-s} \int_{(\frac{\mu l_{\min}}{nU l_{\max}})^2 - 1}^{(\frac{\mu}{nU})^2 - 1} {\rm d}x f(x, M, \Lambda, s, n) + \sum_{n=\frac{l_{\min}\mu}{l_{\max}U}}^{\mu/U} n^{-s} \int_0^{(\frac{\mu}{nU})^2 - 1} {\rm d}x f(x, M, \Lambda, s, n) \quad (104)$$

with

$$f(x) = \frac{2+x}{1+x} [1 + \Lambda x]^{-(2+s)/2} [J_n'(nMx^{1/2})]^2. \quad (105)$$

The expression $W(\Lambda, M, s, U)$ is evaluated in Appendix B. The general calculation is very involved, but for energetic particles with super-Alfvenic ($v \gg V_{\rm A}$) velocities and gyroradii smaller than $R_L < l_{\max}/2\pi$, which is of order 1 pc, we obtain approximations for small ($\epsilon < |\mu| \leq 1/\sqrt{2}$) and large ($1/\sqrt{2} < |\mu| \leq 1$) pitch-angle cosines. In the former region we obtain

$$W_s = W(\Lambda, s, V > 1, \epsilon < |\mu| \leq 1/\sqrt{2}) \simeq \frac{\zeta(s+1)}{\pi M} \times \left(\frac{\pi^{1/2}}{\Lambda^{1/2}} \frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]} + 2[1+\Lambda]^{-(2+s)/2} \times \left[{}_2F_1 \left(1+\frac{s}{2}, 1; \frac{5+s}{2}; \frac{1}{1+\Lambda} \right) + {}_2F_1 \left(1+\frac{s}{2}, 1; \frac{3}{2}; \frac{\Lambda}{1+\Lambda} \right) \right] \right) \quad (106)$$

where we introduced the Riemann zeta-function $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$. For large μ we find the constant value

$$W_l = W(\Lambda, s, V > 1, 1/\sqrt{2} < |\mu| \leq 1) \simeq (2s\Lambda)^{-1} \quad (107)$$

9.3. Gyroradii $R_L < l_{max}/2\pi$

For cosmic ray particle with gyrodii less than $R_{\rm L} < l_{\max}/2\pi$ we obtain with Eqs. (3)–(4) for $I_{\rm tot}$

$$D^G_{\mu\mu}(|\mu| > \epsilon) \simeq \frac{\pi}{2} \frac{(\delta B)^2}{B_0^2} R_L^{s-2} v(1-\mu^2)|\mu|^{s-1} \times \left[\int_{k_{\min}}^{k_{\max}} {\rm d}k\, k^{-s} \right]^{-1} \begin{cases} \frac{W_s}{J(\Lambda)} & \text{for } \epsilon < |\mu| \leq 2^{-1/2} \\ (2s\Lambda J(\Lambda))^{-1} & \text{for } |\mu| > 2^{-1/2} \end{cases} \quad (108)$$

with W_s given in Eq. (106). For isotropic turbulence ($\Lambda = 1$) Eq. (108) is in accord with the approximations (58b) and (58c) of SM, in particular the dependences on cosmic ray particle properties (μ, v, R_L) are identical.

10. Appendix B: Calculation of expression W(Λ, M, s, U)

Obviously, Eq. (104) is equal to

$$W(\Lambda, M, s, U) = \sum_{n=1}^{\mu/U} n^{-s} j_1 - \sum_{n=1}^{\frac{l_{\min}\mu}{l_{\max}U}} n^{-s} j_2 \quad (109)$$

with

$$j_1 = \int_0^{(\frac{\mu}{nU})^2 - 1} {\rm d}x f(x, M, \Lambda, s, n) \quad (110)$$

and

$$j_2 = \int_0^{(\frac{\mu l_{\min}}{nU l_{\max}})^2 - 1} {\rm d}x f(x, M, \Lambda, s, n). \quad (111)$$

To evaluate these two integrals approximately, we will use the approximation (89) for values of $M^2x \geq 1$, while for small values of $M^2x < 1$ we use

$$J_n(nz, z<1) \simeq \frac{n^n z^n}{2^n\Gamma(n+1)} \tag{112}$$

implying

$$[J'_n(nz, z<1)]^2 \simeq \left[\frac{n^n z^{n-1}}{2^n\Gamma(n+1)}\right]^2. \tag{113}$$

We consider both integrals in turn.

10.1. Integral j_1

Here we have to compare the value of M^{-2} with the upper integration boundary of the integral (110). If

$$M^{-2} > \left(\frac{\mu}{nU}\right)^2 - 1 \tag{114}$$

we can use approximation (113) throughout to obtain

$$j_1 \simeq \left[\frac{n^{2n}M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right]\int_0^{(\frac{\mu}{nU})^2-1} \times \mathrm{d}x\, x^{n-1}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2}. \tag{115}$$

In the opposite case $M^{-2} < (\frac{\mu}{nU})^2 - 1$ we use approximation (113) in the range $0 \leq x \leq M^{-2}$ and approximation (89) in the range $M^{-2} < x \leq (\frac{\mu}{nU})^2 - 1$ with the result

$$j_1 \simeq \left[\frac{n^{2n}M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right]\int_0^{M^{-2}} \mathrm{d}x\, x^{n-1}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2} + \frac{1}{\pi n M}\int_{M^{-2}}^{(\frac{\mu}{nU})^2-1} \mathrm{d}x x^{-1/2}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2} \times[1-(-1)^n\sin(2nMx^{1/2})]. \tag{116}$$

The condition (114) translates into

$$\mu^2(1-\mu^2) < (nU)^2. \tag{117}$$

Since the maximum value of the left hand side of this inequality is less than $1/4$ the inequality is always fulfilled for values of $nU \geq 1/2$ implying

$$j_1(nU \geq 1/2) \simeq \left[\frac{n^{2n}M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right] \times \int_0^{(\frac{\mu}{nU})^2-1} \mathrm{d}x\, x^{n-1}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2}. \tag{118}$$

If $nU < 1/2$ the inequality (117) is fulfilled in the pitch angle ranges

$$0 \leq \mu^2 \leq \frac{1}{2}\left[1-\sqrt{1-4n^2U^2}\right];$$

and

$$\frac{1}{2}\left[1+\sqrt{1-4n^2U^2}\right] \leq \mu^2 \leq 1 \tag{119}$$

implying again Eq. (118) in this range.

In the intermediate pitch angle range $\frac{1}{2}[1-\sqrt{1-4n^2U^2}] < \mu^2 < \frac{1}{2}[1+\sqrt{1-4n^2U^2}]$ approximation (116) holds.

10.2. Integral j_2

Here we have to compare the value of M^{-2} with the upper integration boundary of the integral (111). If

$$M^{-2} > \left(\frac{\mu l_{\rm min}}{nUl_{\rm max}}\right)^2 - 1 \tag{120}$$

we can use approximation (113) throughout to obtain

$$j_2 \simeq \left[\frac{n^{2n}M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right]\int_0^{(\frac{\mu l_{\rm min}}{nUl_{\rm max}})^2-1} \mathrm{d}x\, x^{n-1}\frac{2+x}{1+x} \times[1+\Lambda x]^{-(2+s)/2}. \tag{121}$$

In the opposite case $M^{-2} < (\frac{\mu l_{\rm min}}{nUl_{\rm max}})^2 - 1$ we use approximation (113) in the range $0 \leq x \leq M^{-2}$ and approximation (89) in the range $M^{-2} < x \leq \left(\frac{\mu l_{\rm min}}{nUl_{\rm max}}\right)^2 - 1$ with the result

$$j_2 \simeq \left[\frac{n^{2n}M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right]\int_0^{M^{-2}} \mathrm{d}x\, x^{n-1}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2} + \frac{1}{\pi n M}\int_{M^{-2}}^{(\frac{\mu l_{\rm min}}{nUl_{\rm max}})^2-1} \mathrm{d}x\, x^{-1/2}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2} \times[1-(-1)^n\sin(2nMx^{1/2})]. \tag{122}$$

The condition (120) translates into

$$\mu^2(1-\mu^2) < (nUl_{\rm min}/l_{\rm max})^2. \tag{123}$$

Again the maximum value of the left hand side of this inequality is less than $1/4$, so that the inequality is always fulfilled for values of $nUl_{\rm min}/l_{\rm max} \geq 1/2$ implying

$$j_2(nUl_{\rm min}/l_{\rm max} \geq 1/2) \simeq \left[\frac{n^{2n}M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right] \times\int_0^{(\frac{\mu l_{\rm min}}{nUl_{\rm max}})^2-1} \mathrm{d}x\, x^{n-1}\frac{2+x}{1+x}[1+\Lambda x]^{-(2+s)/2}. \tag{124}$$

If $nUl_{\rm min}/l_{\rm max} < 1/2$ the inequality (123) is fulfilled in the pitch angle ranges

$$0 \leq \mu^2 \leq \frac{1}{2}\left[1-\sqrt{1-(2nUl_{\rm min}/l_{\rm max})^2}\right];$$

and

$$\frac{1}{2}\left[1+\sqrt{1-(2nUl_{\rm min}/l_{\rm max})^2}\right] \leq \mu^2 \leq 1 \tag{125}$$

implying again Eq. (124) in this range.

In the intermediate pitch angle range $\frac{1}{2}[1-\sqrt{1-(2nUl_{\rm min}/l_{\rm max})^2}] < \mu^2 < \frac{1}{2}[1+\sqrt{1-(2nUl_{\rm min}/l_{\rm max})^2}]$ approximation (122) holds.

10.3. Case of $k_{max} = \infty$

Because we are concerned with the transport of very energetic particles $v \gg V_A$ we do not lose much generality if we extend the turbulence power spectrum to infinitely large wavenumbers, i.e. $k_{\max} = l_{\min}^{-1} = \infty$. In this case $U = 0$ according to Eq. (99) and

$$\frac{l_{\max}}{l_{\min}} U = V = (k_{\min} R_L)^{-1}. \qquad (126)$$

As a consequence, the general expression (109) simplifies enormously to

$$W(\Lambda, M, s, V) = \sum_{n=1}^{\infty} n^{-s} h_1 - \sum_{n=1}^{|\mu|/V} n^{-s} h_2 \qquad (127)$$

with

$$h_1 = \int_0^{\infty} \mathrm{d}x f(x, M, \Lambda, s, n) \qquad (128)$$

and

$$h_2 = \int_0^{(\frac{|\mu|}{nV})^2 - 1} \mathrm{d}x f(x, M, \Lambda, s, n). \qquad (129)$$

Restricting the analysis to cosmic ray particles with gyroradii $R_L < k_{\min}^{-1} = l_{\max}/2\pi$ which is of order 1 pc, the second sum in Eq. (127) vanishes, and we obtain

$$\begin{aligned} W(\Lambda, M, s, V > 1) &= \sum_{n=1}^{\infty} n^{-s} h_1 \\ &= \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} \mathrm{d}x f(x, M, \Lambda, s, n) \\ &= \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} \mathrm{d}x \frac{2+x}{1+x} [1+\Lambda x]^{-(2+s)/2} [J_n'(nMx^{1/2})]^2. \end{aligned} \qquad (130)$$

Obviously, we obtain with the approximations (89) and (113)

$$h_1 \simeq \left[\frac{n^{2n} M^{2n-2}}{2^{2n}\Gamma^2(n+1)}\right] K_1 + \frac{1}{n\pi M} K_2 \qquad (131)$$

with

$$K_1 = \int_0^{M^{-2}} \mathrm{d}x\ x^{n-1} \frac{2+x}{1+x} [1 + \Lambda x]^{-(2+s)/2} \qquad (132)$$

and

$$K_2 = \int_{M^{-2}}^{\infty} \mathrm{d}x\ x^{-1/2} \frac{2+x}{1+x} [1 + \Lambda x]^{-(2+s)/2} \qquad (133)$$

where we neglected the oscillating part in approximation (113). We consider the cases $M \geq 1$ and $M < 1$ corresponding to $|\mu| \leq 1/\sqrt{2}$ and $|\mu| > 1/\sqrt{2}$, respectively.

10.3.1. Small values of $|\mu| \leq 1/\sqrt{2}$

In this case $M^{-2} \ll 1$ is a small quantity, and the integral (132) is approximately

$$\begin{aligned} K_1 &\simeq 2 \int_0^{M^{-2}} \mathrm{d}x x^{n-1} [1 + \Lambda x]^{-(2+s)/2} \\ &= 2M^{-2n} \int_0^1 \mathrm{d}y \frac{y^{n-1}}{(1 + \Lambda M^{-2} y)^{(2+s)/2}} \\ &= \frac{2}{nM^{2n}} [1 + \Lambda M^{-2}]^{-(2+s)/2} \\ &\quad \times {}_2F_1\left(1 + \frac{s}{2}, 1; n+1; \frac{\Lambda}{\Lambda + M^2}\right) \end{aligned} \qquad (134)$$

in terms of the hypergeometric function. In deriving Eq. (134) we have used the transformation formula

$${}_2F_1(a, b; c; z) = (1 - z)^{-a}\ {}_2F_1\left(a, c - b; c; \frac{z}{z-1}\right) \cdot \qquad (135)$$

Likewise, the integral (133) can be approximated as

$$\begin{aligned} K_2 &\simeq \int_0^{\infty} \mathrm{d}x\, x^{-1/2} \frac{2+x}{1+x} [1 + \Lambda x]^{-(2+s)/2} - 2 \\ &\quad \times \int_0^{M^{-2}} \mathrm{d}x x^{-1/2} [1 + \Lambda x]^{-(2+s)/2} \\ &= \int_0^{\infty} \mathrm{d}x\, x^{-1/2} [1 + \Lambda x]^{-(2+s)/2} \\ &\quad + \int_0^{\infty} \mathrm{d}x\, \frac{x^{-1/2}}{1+x} [1 + \Lambda x]^{-(2+s)/2} \\ &\quad - \frac{4}{M} [1+\Lambda M^{-2}]^{-(2+s)/2}\, {}_2F_1\left(1+\frac{s}{2}, 1; \frac{3}{2}; \frac{\Lambda}{\Lambda+M^2}\right) \\ &\simeq \frac{\pi^{1/2}}{\Lambda^{1/2}} \frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]} - \frac{4}{M} [1 + \Lambda M^{-2}]^{-(2+s)/2} \\ &\quad \times {}_2F_1\left(1 + \frac{s}{2}, 1; \frac{3}{2}; \frac{\Lambda}{\Lambda + M^2}\right) + \int_0^1 \mathrm{d}x x^{-1/2} \\ &\quad \times [1 + \Lambda x]^{-(2+s)/2} + \int_1^{\infty} \mathrm{d}x x^{-3/2} [1 + \Lambda x]^{-(2+s)/2} \\ &= \frac{\pi^{1/2}}{\Lambda^{1/2}} \frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]} - \frac{4}{M} [1+\Lambda M^{-2}]^{-(2+s)/2} \\ &\quad \times {}_2F_1\left(1+\frac{s}{2}, 1; \frac{3}{2}; \frac{\Lambda}{\Lambda+M^2}\right) + 2[1 + \Lambda]^{-(2+s)/2} \\ &\quad \times \left({}_2F_1\left(1 + \frac{s}{2}, 1; \frac{5+s}{2}; \frac{1}{1+\Lambda}\right) \right. \\ &\quad \left. + {}_2F_1\left(1 + \frac{s}{2}, 1; \frac{3}{2}; \frac{\Lambda}{1+\Lambda}\right) \right) \cdot \end{aligned} \qquad (136)$$

Collecting terms in Eq. (131) we obtain to lowest order in the small quanitity $M^{-1} \ll 1$

$$\begin{aligned} h_1 &\simeq \frac{1}{\pi n M} \left(\frac{\pi^{1/2}}{\Lambda^{1/2}} \frac{\Gamma[\frac{1+s}{2}]}{\Gamma[\frac{2+s}{2}]} + 2[1 + \Lambda]^{-(2+s)/2} \right. \\ &\quad \times \left[{}_2F_1\left(1 + \frac{s}{2}, 1; \frac{5+s}{2}; \frac{1}{1+\Lambda}\right) \right. \\ &\quad \left.\left. + {}_2F_1\left(1 + \frac{s}{2}, 1; \frac{3}{2}; \frac{\Lambda}{1+\Lambda}\right) \right] \right) \cdot \end{aligned} \qquad (137)$$

10.3.2. Large values of $|\mu| > 1/\sqrt{2}$

Here $M^{-2} \gg 1$ is large, so that the integral (133) is approximately

$$K_2 \simeq \int_{M^{-2}}^{\infty} \mathrm{d}x\, x^{-1/2}[1+\Lambda x]^{-(2+s)/2} = M^{-1}\int_0^1 \mathrm{d}t\, t^{(s-1)/2}[t+\Lambda M^{-2}]^{-(2+s)/2} \quad (138)$$

where we substituted $x = (M^2 t)^{-1}$. In terms of hypergeometric functions we obtain

$$K_2 \simeq \frac{2}{s+1} M^{-1}[\frac{M^2}{\Lambda+M^2}]^{(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s+3}{2}; \frac{M^2}{\Lambda+M^2}\right). \quad (139)$$

Likewise, the integral (132) can be approximated as

$$K_1 \simeq \int_0^{M^{-2}} \mathrm{d}x\, x^{n-1}[1+\Lambda x]^{-(2+s)/2} - \int_{M^{-2}}^{\infty} \mathrm{d}x\, x^{n-2}[1+\Lambda x]^{-(2+s)/2} + \int_0^1 \mathrm{d}x\, x^{n-1}[1+\Lambda x]^{-(2+s)/2} + \int_1^{\infty} \mathrm{d}x\, x^{n-2}[1+\Lambda x]^{-(2+s)/2}$$
$$= M^{-2n}\int_0^1 \mathrm{d}t\, t^{n-1}[1+\Lambda M^{-2}t]^{-(2+s)/2} - M^{4+s-2n}\Lambda^{-(2+s)/2}\int_0^1 \mathrm{d}t t^{(s+2-2n)/2} \times[1+\Lambda^{-1}M^2 t]^{-(2+s)/2} + \frac{1}{n}[1+\Lambda]^{-(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{\Lambda+1}\right) + \Lambda^{-(2+s)/2} \times \int_0^1 \mathrm{d}t\, t^{(s+2-2n)/2}[1+\Lambda^{-1}t]^{-(2+s)/2}. \quad (140)$$

In terms of hypergeometric functions we obtain

$$K_1 \simeq \frac{1}{n}M^{2+s-2n}[\Lambda+M^2]^{-(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{\Lambda+M^2}\right) + \frac{1}{n}[1+\Lambda]^{-(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{1+\Lambda}\right) + \frac{2}{s+4-2n} \times[1+\Lambda]^{-(2+s)/2}\, {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s}{2}+3-n; \frac{1}{1+\Lambda}\right) - \frac{2}{s+4-2n}M^{4+s-2n}[\Lambda+M^2]^{-(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s}{2}+3-n; \frac{M^2}{\Lambda+M^2}\right). \quad (141)$$

Collecting terms in Eq. (131) we obtain

$$h_1 \simeq \frac{2}{(s+1)n\pi M^2}\left[\frac{M^2}{\Lambda+M^2}\right]^{(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s+3}{2}; \frac{M^2}{\Lambda+M^2}\right) + \frac{n^{2n}}{2^{2n}\Gamma^2[n+1]M^2} \times\left(\frac{1}{n}\left[\left[\frac{M^2}{\Lambda+M^2}\right]^{(2+s)/2} {}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{\Lambda+M^2}\right) + M^{2n}[1+\Lambda]^{-(2+s)/2}{}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{1+\Lambda}\right)\right] + \frac{2}{s+4-2n}\left[M^{2n}[1+\Lambda]^{-(2+s)/2} \times {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s}{2}+3-n; \frac{1}{1+\Lambda}\right) - M^{4+s} \times[\Lambda+M^2]^{-(2+s)/2}{}_2F_1\left(1+\frac{s}{2}, 1; \frac{s}{2}+3-n; \frac{M^2}{\Lambda+M^2}\right)\right]\right). \quad (142)$$

Because $M^2 \ll 1$ the leading terms of Eq. (142) are

$$h_1 \simeq \frac{1}{nM^2}\left[\frac{M^2}{\Lambda+M^2}\right]^{(2+s)/2}\left[\frac{2}{(s+1)\pi} \times {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s+3}{2}; \frac{M^2}{\Lambda+M^2}\right) + \frac{n^{2n}}{2^{2n}\Gamma^2[n+1]} \times {}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{\Lambda+M^2}\right)\right]. \quad (143)$$

For $M^2 \ll 1$ the two hypergeometric functions in Eq. (143) can be approximated by

$${}_2F_1\left(1+\frac{s}{2}, 1; \frac{s+3}{2}; \frac{M^2}{\Lambda+M^2}\right) \simeq {}_2F_1\left(1+\frac{s}{2}, 1; \frac{s+3}{2}; 0\right) = 1 \quad (144)$$

and for $n \geq 2$

$${}_2F_1\left(1+\frac{s}{2}, 1; n+1; \frac{\Lambda}{\Lambda+M^2}\right) \simeq {}_2F_1(1+\frac{s}{2}, 1; n+1; 1) = \frac{2n}{2n-2-s} \quad (145)$$

yielding factors of order unity, whereas for $n = 1$

$${}_2F_1\left(1+\frac{s}{2}, 1; 2; \frac{\Lambda}{\Lambda+M^2}\right) = \int_0^1 \mathrm{d}t\left[1-\frac{\Lambda}{\Lambda+M^2}t\right]^{-(2+s)/2} = \frac{2}{s}\frac{\Lambda+M^2}{\Lambda}\left[\left(\frac{\Lambda+M^2}{M^2}\right)^{s/2} - 1\right] \simeq \frac{2}{s}\frac{(\Lambda+M^2)^{(2+s)/2}}{\Lambda M^s} \quad (146)$$

which, because of the $\propto M^{-s} \gg 1$ dependence, dominates the bracket of Eq. (143). Consequently, we obtain

$$h_1 \simeq (2s\Lambda)^{-1}. \quad (147)$$

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