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Hidden problems of thin-flux-tube approximation

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Abstract. This paper scrutinizes the validity of the thin-flux-tube approximation for magnetic flux tubes embedded in a surrounding magnetic plasma. It is shown that the thin-flux-tube approximation gives an accurate description of surface waves for $C_{\rm T}^2 > C_{\rm Ae}^2$, body waves for $C_{\rm Te}^2 > C_{\rm T}^2$ and surface leaky waves for $C_{\rm Te}^2 < C_{\rm T}^2 < C_{\rm Ae}^2$. The Leibovich-Roberts equation for nonlinear surface waves in a flux tube embedded in a field free plasma is generalized to a flux tube immersed in a magnetic plasma. The generalized Leibovich-Roberts (GLR) equation describes the propagation of nonlinear slow surface, body and surface leaky waves in tubes. The shortcomings of the GLR equation are discussed. The Korteweg-de Vries equation (KdV) is generalized for surface waves. The advantage of the second order thin-flux-tube approximation is shown. Two scenarios for the heating of coronal loops are discussed. It is emphasized, that the application of the thin-flux-tube approximation to thin tubes of non-zero diameter has to take into account possible wave emission by the tube and shock front formation for amplitudes in excess of some critical value.

Key words. agnetohydrodynamics (MHD) – waves – Sun: corona – oscillations

1. Introduction

The study of the dynamics of magnetic solar and stellar atmospheres is severely complicated by the joint presence of stratification due to gravity and structuring due to the inhomogeneous magnetic field. Part of this complication can be circumvented by using the thin-flux-tube approximation introduced by Roberts & Webb (1978). This approximation is often used for studying MHD waves in magnetic flux tubes. A major advantage of this approximation is that a 3D-problem is reduced to a set of 1D equations. This makes a wide variety of problems of linear and nonlinear waves in the solar atmosphere tractable for theoretical analysis and numerical simulation. However, when approximations are introduced, it is necessary to try and understand when and where they give an accurate description of physical reality and to see whether or not essential effects have been omitted. Naively one might assume that $kR_0 \ll 1$ (k is the longitudinal wavenumber and R_0 is the tube radius) is the only condition to be satisfied for the thin-flux-tube approximation to be valid. The aim of the current paper is to show that the conditions for the thin-flux-tube approximation to be valid are more subtle and are related to the effects of the surrounding plasma on the tube dynamics.

Roberts & Webb (1978) showed that the thin-flux-tube approximation describes slow surface modes. Ferriz-Mas et al. (1989) reconsidered the problem and confirmed that the thin-flux-tube approximation describes slow surface modes. Both investigations dealt with waves in an isolated flux tube embedded in a magnetic field free plasma. However, the thin flux tube approximation is also used for studying waves in magnetic flux tubes in the solar chromosphere and corona, where the tubes are surrounded by a magnetic plasma (Herboldt et al. 1985; Ulmschneider et al. 1991). The free boundary condition, which is used in the thin-flux-tube approximation, assumes just constancy of the external pressure without specifying whether this is gas pressure or magnetic pressure or both. As a matter of fact the thin-flux-tube approximation is valid only for an infinitely thin tube, when $kR_0 \to 0$, while it is used for thin tubes with $kR_0 \ll 1$. In that case, the effect of the surroundings depends on the the parameters of the surrounding plasma. In addition, so far the thin-flux-tube limit has only shown to exist for a magnetic field-free external plasma. As a consequence, the conditions for the thin-tube approximation to be valid have not yet been explored for tubes embedded in a magnetic plasma. We focus here on this problem for both linear and nonlinear waves in a thin tube, which sometimes will be referred to as tube waves.

The paper is organized as follows. First, the dispersion equation for waves in flux tubes is rewritten to

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separate the dispersion equation for slow waves. The thin tube limit is explored for a thin tube surrounded by a magnetic plasma. The approximate dispersion relation for tube waves in thin tubes of non-zero diameter is obtained. The advantage of the second order thin-flux-tube approximation (Zhugzhda 1996) is discussed shortly. Weakly nonlinear waves in thin tubes are studied and the Leibovich-Roberts equation is generalized. The KdV equation for slow body waves in a thin tube (Zhugzhda & Nakariakov 1997a) is generalized to surface waves. Finally, two scenarios for the heating of coronal loops by slow waves are discussed using the current analysis of linear and nonlinear waves in thin tubes.

2. Thin flux tube limit

In the thin-flux-tube approximation (Roberts & Webb 1978) the full set of non-linear MHD equations is reduced to the following set of equations:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B}\right)+\frac{\partial}{\partial z}\left(\frac{\rho v}{B}\right)=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{2}$$

$$\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial z}-\frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial z}\right)=0, \tag{3}$$

$$p+\frac{B^2}{8\pi}=p_{\rm ext}, \tag{4}$$

$$BA=\text{constant}, \tag{5}$$

$B(z,t)$ is the longitudinal component of the magnetic field, $v(z,t)$ is the longitudinal component of the flow speed, $A(z,t)$ is cross-section of the tube and $\rho(z,t)$ is the plasma density. The plasma pressure $p(z,t)$ and magnetic pressure $B^2(z,t)/8\pi$ within the flux tube are in lateral balance with the surroundings at pressure $p_{\rm ext}=const.$ on the tube boundary. This set of equations is valid for an infinitely thin flux tube. It describes the so-called linear tube waves, which propagate with the tube velocity $C_{\rm T}=C_{\rm A}C_{\rm S}/\sqrt{C_{\rm A}^2+C_{\rm S}^2}$ and are not dispersive. The thin-flux-tube approximation assumes that the tube is infinitely thin. However, in applications the thin-flux-tube approximation is used to thin tubes with a non-zero diameter. The problems of the thin-flux-tube approximation appear because it neglects the reaction of the external medium: to close the system the external pressure $p_{\rm ext}$ is assumed to be constant.

Before proceeding to hidden problems of the thin-flux-tube approximation we first explore in this section the crucial point whether the thin tube limit exists for an arbitrary external medium. Only then does it make sense to look at possible inaccuracies introduced by using the thin-flux-tube approximation to study waves in thin tubes of non-zero diameter.

2.1. Dispersion equation for slow modes

To attack the problem we modify the dispersion equation for axisymmetric waves in a flux tube immersed in a magnetic plasma, which was derived by (Meerson et al. 1978; Wilson 1980; Spruit 1982; Edwin & Roberts 1983). In what follows we refer to the paper by Edwin & Roberts (1983), as it contains the most complete exploration of the rather complex array of wave modes governed by this equation. For body waves it reads

$$\rho_0(k^2C_{\rm A}^2-\omega^2)m_{\rm e}\frac{K_1(m_{\rm e}R_0)}{K_0(m_{\rm e}R_0)} -\rho_{0\rm e}(k^2C_{\rm Ae}^2-\omega^2)n_o\frac{J_1(n_0R_0)}{J_0(n_0R_0)}=0, \tag{6}$$

where $n_0,\ m_{\rm e}$ are

$$n_0^2=-\frac{(k^2C_{\rm S}^2-\omega^2)(k^2C_{\rm A}^2-\omega^2)}{(C_{\rm S}^2+C_{\rm A}^2)(k^2C_{\rm T}^2-\omega^2)}>0,$$

$$m_{\rm e}^2=\frac{(k^2C_{\rm Se}^2-\omega^2)(k^2C_{\rm Ae}^2-\omega^2)}{(C_{\rm Se}^2+C_{\rm Ae}^2)(k^2C_{\rm Te}^2-\omega^2)}>0,$$

where $C_{\rm S}$, $C_{\rm A}$, $C_{\rm T}$ are sound, Alfvén and tube speeds. The subscript "e" denotes parameters outside the tube. The condition of pressure balance at the interface of the tube with the external plasma limits the choice of the values of the magnetic field and the temperature inside and outside the tube. Let us introduce the dimensionless quantities β,δ and Δ defined as

$$\beta=\frac{C_{\rm S}^2}{C_{\rm A}^2},\quad \beta_{\rm e}=\frac{C_{\rm Se}^2}{C_{\rm Ae}^2},\quad \delta=\frac{C_{\rm Se}^2}{C_{\rm S}^2},$$
$$\Delta=\frac{\rho_{0\rm e}}{\rho_0}=\frac{\beta_{\rm e}}{\delta\beta}\frac{2\beta+\gamma}{2\beta_{\rm e}+\gamma}, \tag{7}$$

and the dimensionless variables x,Ω and j as

$$x=kR_0,\quad \Omega=\frac{\omega}{kC_{\rm T}},\quad j^2=n_0^2R_0^2. \tag{8}$$

The variable Ω can be interpreted either as dimensionless frequency or as dimensionless phase velocity, because $\Omega=C(k)/C_{\rm T}$, where $C(k)=\omega/k$. Both terms are used in the paper.

In order to separate the equation for the slow body waves we rewrite the dispersion Eq. (6) as

$$\left(\frac{1+\beta}{\beta\Omega_{\rm b}^2}-1\right)m_{\rm b}x\frac{K_1(m_{\rm b}x)}{K_0(m_{\rm b}x)} -\left(\frac{(1+\beta)\delta}{\beta_{\rm e}\Omega_{\rm b}^2}-1\right)\Delta j\frac{J_1(j)}{J_0(j)}=0, \tag{9}$$

where

$$m_{\rm b}=\sqrt{\frac{[\delta(1+\beta)-\Omega_{\rm b}^2][\delta(1+\beta)-\beta_{\rm e}\Omega_{\rm b}^2]}{\delta(1+\beta)[\delta(1+\beta)-(1+\beta_{\rm e})\Omega_{\rm b}^2]}}, \tag{10}$$

$$\Omega_{\rm b}^2=\frac{2}{1+\sqrt{1-4\beta x^2a^{-1}}},\ a=(1+\beta)^2(x^2+j^2). \tag{11}$$

The separation of the slow from the fast waves is possible due to choice of the sign in front of the square root in the expression for the frequency Ω_b. When β, β_e, δ and x are fixed, Eq. (9) is an equation with respect to j, and, in general, $j = j(x, \beta, \beta_\mathrm{e}, \delta)$. The frequency of the slow body waves as a function of the dimensionless wavenumber x for fixed β, β_e and δ is defined by (11) after substitution of the root j of Eq. (9). The dispersion equation for fast body waves can be obtained by a change of the sign in front of the square root in (11).

For the surface waves the dispersion equation (Roberts & Edwin 1983) reads

$$\rho_0(k^2C_\mathrm{A}^2 - \omega^2)m_\mathrm{e}\frac{K_1(m_\mathrm{e}R_0)}{K_0(m_\mathrm{e}R_0)} + \rho_{0\mathrm{e}}(k^2C_\mathrm{Ae} - \omega^2)m_\mathrm{o}\frac{I_1(m_0R_0)}{I_0(m_0R_0)} = 0, \quad (12)$$

where $m_0^2 = -n_0^2$. After the introduction of the same dimensionless variables as for body waves except for

$$j^2 = m_0^2R_0^2 = -n_0^2R_0^2, \quad (13)$$

where $n_0^2 < 0$, Eq. (12) is rewritten to separate the dispersion equation for slow surface waves

$$\left(\frac{1+\beta}{\beta\Omega_\mathrm{s}^2} - 1\right) m_\mathrm{s}x\frac{K_1(m_\mathrm{s}x)}{K_0(m_\mathrm{s}x)} + \left(\frac{(1+\beta)\delta}{\beta_\mathrm{e}\Omega_\mathrm{s}^2} - 1\right)\Delta j\frac{I_1(j)}{I_0(j)} = 0, \quad (14)$$

where

$$m_\mathrm{s} = \sqrt{\frac{[\delta(1+\beta) - \Omega_\mathrm{s}^2][\delta(1+\beta) - \beta_\mathrm{e}\Omega_\mathrm{s}^2]}{\delta(1+\beta)[\delta(1+\beta) - (1+\beta_\mathrm{e})\Omega_\mathrm{s}^2]}}, \quad (15)$$

$$\Omega_\mathrm{s}^2 = \frac{2}{1 + \sqrt{1 - 4\beta x^2 a^{-1}}}, a = (1+\beta)^2(x^2 - j^2). \quad (16)$$

As Eqs. (9), (14) can be considered as an equation for j, when β, β_e, δ and x are fixed. The frequency of the surface waves as a function of the dimensionless wavenumber x is defined by (16) after substitution of the root $j(x, \beta, \beta_\mathrm{e}, \delta)$ of Eq. (14). The dispersion equation for slow surface waves is separated from that for fast waves due to choice of the sign in front of square root in the expression for Ω_s (16).

2.2. Thin flux tubes immersed in plasma with a "weak" magnetic field

In most cases the flux tubes in the photospheric and subphotospheric layers are isolated from each other by magnetic field free plasmas with $C_\mathrm{Ae} = 0$ and $\beta \lessapprox 1$. It is our impression that Roberts & Webb (1978) had this situation in mind, when they studied the thin tube limit. We consider this situation first to explain our approach, which differs from that of Roberts & Webb (1978). Figure 2 shows the results of numerically solving the dispersion Eq. (14) for slow body (9) and slow surface (14) modes for $\beta = 1$, $\beta_\mathrm{e}^{-1} = 0$ and for different ratios of external and internal sound velocities $\delta = 1, 2, 5, 10, 20$. The first root of the dispersion Eq. (14) for slow surface modes tends

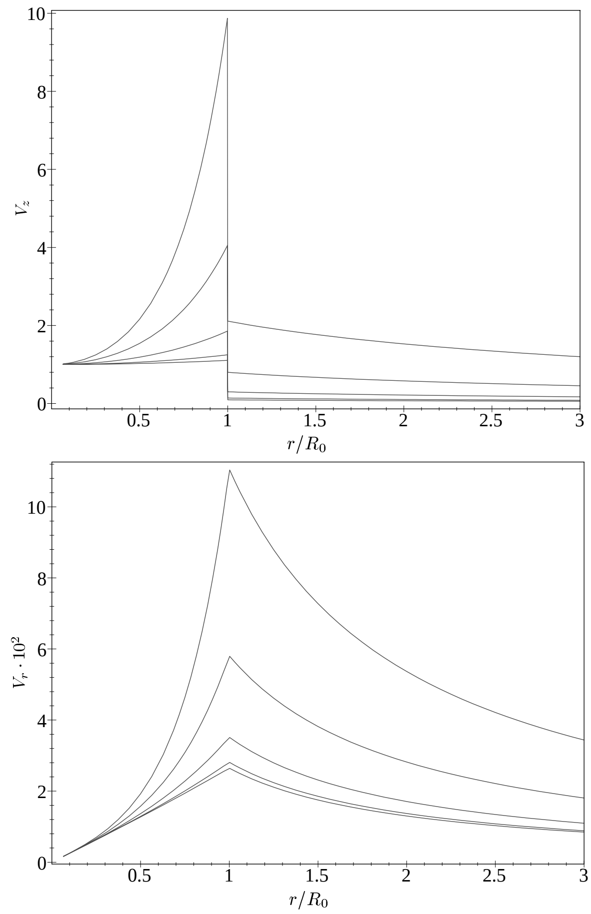

Fig. 1. The eigenfunctions of longitudinal velocity $V_\mathrm{z}(r/R_0)$ (top) and radial velocity $V_\mathrm{r}(r/R_0)$ (bottom) for $kR_0 = 0.1$, $\beta = 1$, $\beta_\mathrm{e}^{-1} = 0$ and $\delta = 1$ (bottom curves), $2, 5, 10, 15$ (top curves) for slow surface mode.

to zero, when we decrease the diameter of the tube for a fixed wavenumber ($R_0 \to 0$, $kR_0 \to 0$) and for all values of the ratio of the external and internal sound speeds δ. It is well-known (Edwin & Roberts 1983) that the eigenfunctions of the wave quantities in a flux tube can be expressed in terms of Bessel functions. Here the eigenfunctions of $V_\mathrm{z}(r)$ and $V_\mathrm{r}(r)$ are proportional to $J_0\,(j_1r/R_0)$ and $J_1\,(j_1r/R_0)$ respectively. When $j_1 \to 0$ the longitudinal velocity tends to a non-zero constant across the tube, while the radial velocity tends to zero. Results of exact calculations for the eigenfunctions of the surface mode in a thin flux tube ($kR_0 = 0.1$) are shown in Fig. 1. The amplitude of the longitudinal velocity is almost constant across the tube for $\delta = 1, 2, 5$, while it increases toward the boundary for $\delta = 10, 15$. The radial velocity is one order of magnitude smaller. When we decrease the diameter of the tube the longitudinal velocity tends more and more to a constant across the tube. The eigenfunctions of the slow mode show that the accuracy of the thin-flux-tube approximation for describing the surface slow modes in thin flux tubes depends on the value of kR_0. Hence, the thin tube limit exists, because there is a root of the dispersion Eq. (14), which tends to zero with $kR_0 \to 0$

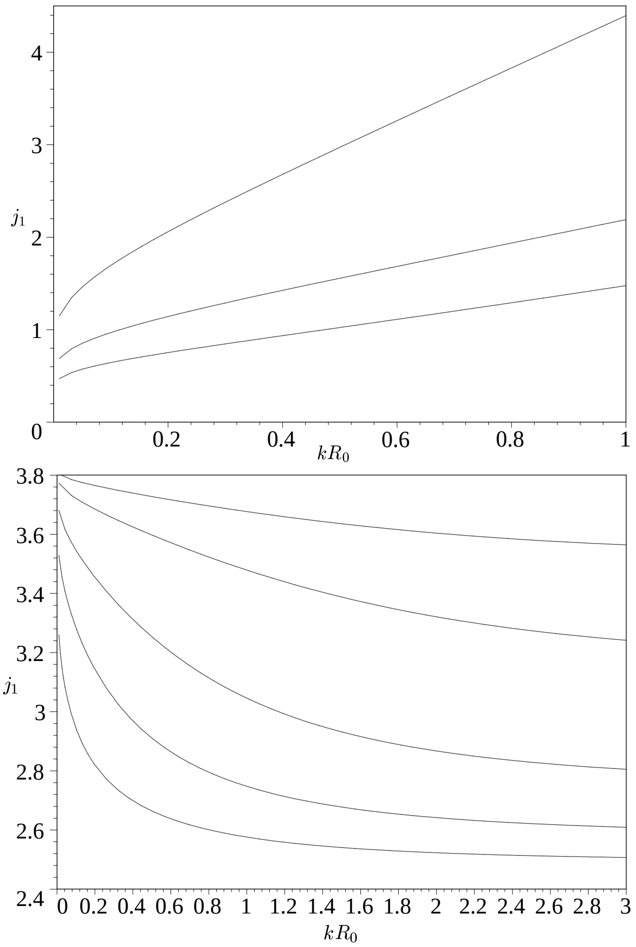

Fig. 2. The dependence of the first root j_1 of dispersion equations for slow surface (14) (top) and slow body (9) (bottom) modes on kR_0 for $\beta = 1, \beta_e^{-1} = 0$ and $\delta = 1, 2, 5, 10, 20$ (bottom curve) for body waves and $\delta = 1, 2, 5$ (top curve) for surface waves.

(see, Fig. 2). In summary, the present analysis shows, that the thin-flux-tube approximation gives a correct description of the slow surface modes for a flux tube embedded in a magnetic free plasma. We arrive at the same conclusion as Roberts & Webb (1978). It is straightforward to check, whether there is a similar root of the dispersion Eq. (14) for surface waves when $\beta_e^{-1} \neq 0$. The dispersion Eq. (14) has a root, which tends to zero for $kR_0 \to 0$, when the condition

$$\beta_e > \frac{(1+\beta)\delta}{\Omega_s^2} \tag{17}$$

is satisfied. When this condition is violated, the first term in the dispersion Eq. (14) reverses sign and the dispersion equation has not any real roots. Thus, the conclusion by Roberts & Webb (1978), that the thin-flux-tube approximation describes slow surface waves is only valid when the external magnetic field is sufficiently small so that condition (17) is satisfied. For the thin flux limit $kR_0 \to 0$, when $\Omega_s \to 1$ (the detailed analysis of the limit is

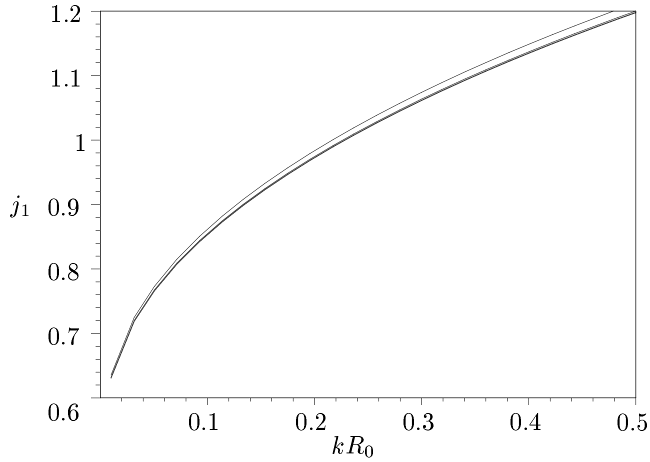

Fig. 3. The dependence of the first root j_1 of dispersion Eq. (9) for slow body mode on kR_0 for $\beta = 0.01$, $\beta_e = 0.009$ and $\delta = 1.1$ top curve, $2, 5, 10, 20$.

presented in Sect. 3, the condition (17) can be replaced by the approximate condition

$$\beta_e > (1+\beta)\delta, \tag{18}$$

which reads in dimension variables

$$C_T^2 > C_{Ae}^2. \tag{19}$$

2.2.1. Slow body waves

It is instructive to explore the dispersion Eq. (9) for slow body waves, when the condition (18) is valid. First of all Eq. (9) has an infinite number of real roots, while Eq. (14) has only one real root. Figure 2 shows numerical solutions to Eq. (9). The first root of the dispersion Eq. (9) of the slow body modes varies in the range $2.405 < j_1 < 3.83$ and in the limit $kR_0 \to 0$ it tends to $j_1 = 3.83$ for all values of δ. The behaviour of the first root for the general case of $\beta_e^{-1} \neq 0$ is the same. The eigenfunction of the longitudinal velocity of the body mode changes sign inside the tube and does not become constant across the tube, as it should do according to the thin flux tube approximation. In addition, the eigenfunction of the radial velocity does not tend to zero, and this also is in contradiction with the assumption of a free boundary in the thin-flux-tube approximation. Thus, the thin-flux-tube approximation can not describe slow body modes for the case of (18).

2.3. Flux tubes immersed in a plasma with a "strong" magnetic field

Flux tubes in the chromosphere and corona cannot be embedded in a field free plasma, because the gas pressure is small in comparison to the magnetic pressure. Thus, the flux tubes in the chromosphere and corona differ from the surrounding plasma in density and/or temperature. Also, in the case of the solar corona and chromosphere the name flux tubes is rather misleading, as the magnetic field can be weaker inside the structure than outside since the gas

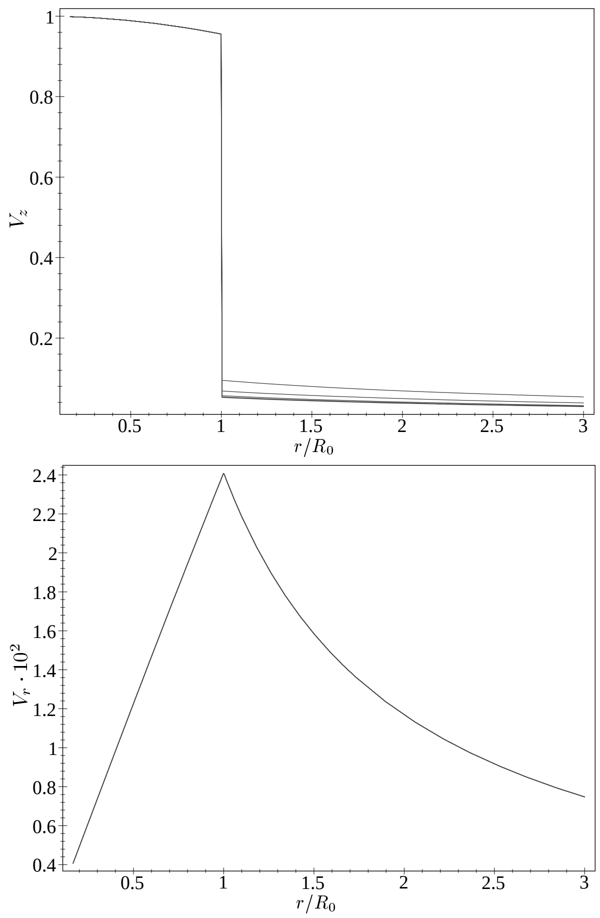

Fig. 4. The eigenfunctions of longitudinal velocity $V_{\rm z}(r/R_0)$ (top) and radial velocity $V_{\rm r}(r/R_0)$ (bottom) for $kR_0 = 0.1$, $\beta = 1$, $\beta_{\rm e} = 0.009$ and $\delta = 1.1$ (top curves), $2, 5, 10, 20$ (bottom curves) for slow body mode.

pressure is lower outside due to the temperature or density difference. Although the gas pressure is negligible in comparison with the magnetic pressure in the corona, it is essential for the properties of waves in tubes. The equilibrium magnetic configuration in the corona can be considered as a force-free. The thin-flux tube approximation for swaying tubes is used to explore the dynamics of flux tubes in the chromosphere and corona (Herboldt et al. 1985; Ulmschneider et al. 1991). The longitudinal waves in a swaying tube obey the same equation as for straight tubes except for the effect of the nonlinear coupling with the kink mode (Nakariakov et al. 1996).

Let us explore the dispersion Eq. (9) for slow body waves, when the condition

$$\beta_{\rm e} < \frac{(1+\beta)\delta}{\Omega_{\rm b}^2} - 1 \tag{20}$$

is satisfied. In this case the real roots of the dispersion Eq. (9) exist only if the Bessel functions J_0 and J_1 are of the same sign, and that is possible when $0 \le j_1 < 2.4$. Figure 3 shows how the first root of the dispersion Eq. (9) for slow body waves depends on kR_0 for $\beta = 0.01$ and $\beta_{\rm e} = 0.009$ and for different values of δ, corresponding to conditions in coronal loops. The first root of (9) tends to zero $j_1 \rightarrow 0$ for all values of δ. This means that the eigenfunctions of the longitudinal and transversal velocity tend to a constant and zero respectively, as for surface modes in a tube embedded in an non-magnetic plasma. Figure 4 shows eigenfunctions of the slow body mode for a thin tube with $kR_0 = 0.1$. The eigenvalues depend only slightly on the value of δ. The eigenfunction of the longitudinal velocity is almost constant inside the tube. The longitudinal velocity is smaller close to the boundary of the tube, while for a tube embedded in a plasma with a "weak" magnetic field, it increases towards the tube boundary. This is because the tube mode in a thin tube with a "weak" field is a slow surface wave, while for a "strong" external magnetic field the tube mode is a slow body wave. The amplitude of transverse velocity is about two orders smaller than that of the longitudinal velocity. The dispersion Eq. (14) for the surface modes has not any solution at all when the tube is embedded in a plasma with a "strong" magnetic field. Thus, surface waves do not exist then and the thin flux tube approximation describes the dynamics of the slow body modes, when the condition (20) is satisfied. For rather thin tubes, when $kR_0 \rightarrow 0$, the condition can be replaced by the approximate condition

$$\beta_{\rm e} < (1+\beta)\delta - 1, \tag{21}$$

which in dimensional variables reads

$$C_{\rm Te}^2 > C_{\rm T}^2. \tag{22}$$

The effect of the surrounding plasma is crucial for the properties of the body waves. The sets of roots of the dispersion Eq. (9) for slow body waves for a "weak" and a "strong" external magnetic field are very different. For a "weak" field the roots of (9) vary with x in the ranges $j_{0,n} < j_n < j_{1,n}$, where $j_{0,n}$ are roots of the equation $J_0(j) = 0$ and $j_{1,n}$ roots of the equations $J_1(j) = 0$. For a "strong" external field the first root of (9) varies in the range $0 < j_1 < j_{0,1}$; the next roots for $n > 1$ are in the ranges $j_{1,n-1} < j_n < j_{0,n+1}$. For a "strong" field the longitudinal velocity of the first mode ($n = 1$) does not change sign inside the tube (see, Fig. 4). There is no similar body mode for a "weak" magnetic field. Thus, the first body mode for a "strong" field only exists if the effect of surrounding plasma is taken into account. Up till now there was confusion in this matter. Ferriz-Maz et al. (1989) claimed, that the thin-flux tube approximation describes surface waves but does not describe slow body waves. The confusion is due to the fact that the analysis by (Ferriz-Maz et al. 1989) was restricted to tubes immersed in a plasma free of magnetic field. The statement by Ferriz-Maz et al. (1989) is not correct, when condition (21) is valid. Moreover, in this case surface waves do not exist at all.

2.4. Flux tubes immersed in plasma with an "intermediate" magnetic field

In the previous sections we showed that for "weak" and "strong" external fields defined by the conditions (17) and (20) the thin-flux-tube approximation describes slow surface or slow body waves, respectively. But in the range defined by the inequalities

$$(1+\beta)\delta - 1 < \beta_{\rm e} < (1+\beta)\delta, \tag{23}$$

it is not clear whether the thin tube limit makes sense, and if it does, which wave modes it describes accurately. This condition in dimensional variables (8) reads

$$C_{\rm Te}^2 < C_{\rm T}^2 < C_{\rm Ae}^2. \tag{24}$$

The conditions ((23), (24)) are valid in the limit $kR_0 \to 0$ as it follows from the limiting conditions (18) and (21), while the exact condition for "intermediate" magnetic fields is

$$\frac{(1+\beta)\delta}{\Omega_{\rm b}} - 1 < \beta_{\rm e} < \frac{(1+\beta)\delta}{\Omega_{\rm s}}. \tag{25}$$

In this range of parameters only leaky surface and body waves are possible, because the dispersion Eqs. ((14), (9)) do not have any real roots. To obtain the dispersion equation for the leaky slow waves the modified Bessel functions K_0 and K_1 in the dispersion Eqs. ((6), (12)) have to be replaced by the Hankel functions $H_0^{(2)}$ and $H_1^{(2)}$ respectively, as the Hankel functions describe waves outgoing from the tube. For slow leaky surface waves the dispersion Eq. (12) in the dimensionless variables (23) is

$$\left(\frac{1+\beta}{\beta\Omega_{\rm s}^2} - 1\right) n_{\rm s} x \frac{H_1^{(2)}(n_{\rm s}x)}{H_0^{(2)}(n_{\rm s}x)} - \left(\frac{(1+\beta)\delta}{\beta_{\rm e}\Omega_{\rm s}^2} - 1\right)\Delta j \frac{I_1(j)}{I_0(j)} = 0, \tag{26}$$

where

$$n_{\rm s} = \sqrt{\frac{[\Omega_{\rm s}^2 - \delta(1+\beta)][\delta(1+\beta) - \beta_{\rm e}\Omega_{\rm s}^2]}{\delta(1+\beta)[\delta(1+\beta) - (1+\beta_{\rm e})\Omega_{\rm s}^2]}} \tag{27}$$

and a and $\Omega_{\rm s}$ are defined by (16). Figure 5 shows how the real and imaginary parts of the root of the dispersion Eq. (26) depend on kR_0. Both the real and imaginary part of the root go to zero. Consequently, the longitudinal velocity approaches a constant absolute value across the tube, while the radial velocity tends to zero. A striking characteristic of the eigenfunctions of the leaky waves is that they are complex due to wave emission by the tube. Thus, the thin tube limit exists for a "intermediate" magnetic field outside the tube (23). The thin-flux-tube approximation describes leaky slow surface waves, when the condition (23) is valid. In addition to the complex roots corresponding to the leaky slow surface waves just discussed, the dispersion relation (26) also has complex roots which correspond to the leaky slow body waves. They do not tend to zero with $kR_0 \to 0$. Thus, the roots of the slow surface dispersion relation increase in number from one real root to an infinite number of complex roots when passing from the dispersion relation (14) for classical surface waves to the relation for the leaky surface waves (26). The dispersion Eq. (6) for the slow body waves can also be rewritten for outgoing waves by replacing the modified Bessel functions by Hankel functions

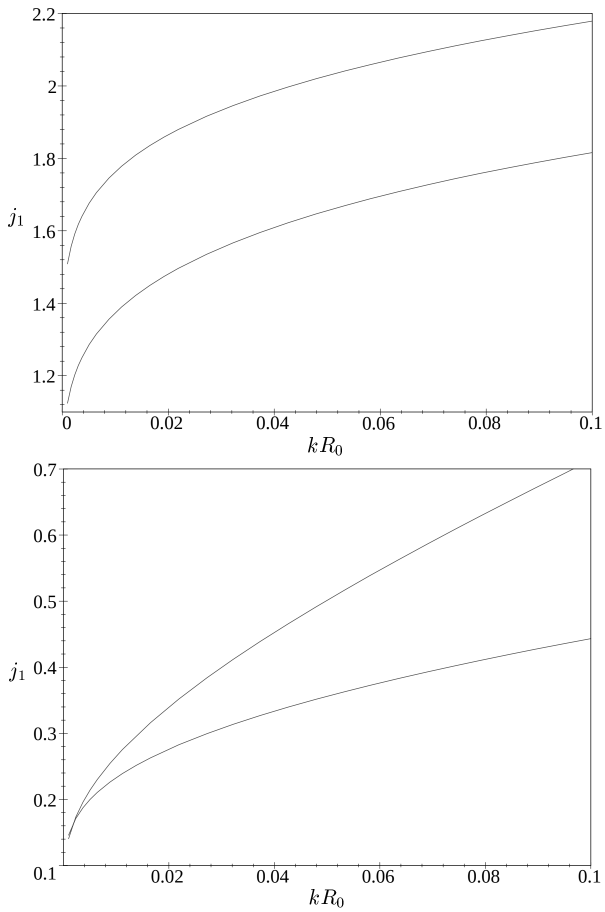

Fig. 5. The dependence of the real (top) and imaginary (bottom) parts of the first root j_1 of dispersion Eq. (26) for leaky surface mode (top) on $x = kR_0$ for $\beta = 0.01$, $\beta_{\rm e} = 0.8$ and $\delta = 1$ (top curves), 1.7 (bottom curves).

$$\left(\frac{1+\beta}{\beta\Omega_{\rm b}^2} - 1\right) n_{\rm b} x \frac{H_1^{(2)}(n_{\rm b}x)}{H_0^{(2)}(n_{\rm b}x)} + \left(\frac{(1+\beta)\delta}{\beta_{\rm e}\Omega_{\rm b}^2} - 1\right)\Delta j \frac{J_1(j)}{J_0(j)} = 0, \tag{28}$$

where

$$n_{\rm b} = \sqrt{\frac{[\Omega_{\rm b}^2 - \delta(1+\beta)][\delta(1+\beta) - \beta_{\rm e}\Omega_{\rm b}^2]}{\delta(1+\beta)[\delta(1+\beta) - (1+\beta_{\rm e})\Omega_{\rm b}^2]}}, \tag{29}$$

and a and $\Omega_{\rm b}$ are defined by (11). This dispersion equation, which has to be considered as the dispersion equation for leaky slow body waves, has complex roots, which differ from the solution of (26) only by the factor i. This comes as no surprise, because the Bessel functions satisfy the condition $I(x) = i^{-n} J_n(ix)$ which makes it possible to transform Eq. (26) into Eq. (28) by the substitution $j \rightarrow ij$. Comparison of the expressions for the frequencies of the surface and body waves (16) and (11) shows, that $\Omega_{\rm b}^2(ij) = \Omega_{\rm s}^2(j)$. Therefore the roots of Eqs. ((26), (28)) lead to the same set of wave mode frequencies. Thus the dispersion Eqs. ((26), (28)) are just two versions of the same dispersion equation for slow waves. At this point we face the problem of how to distinguish between leaky surface waves and body waves? The differences in the wave functions, which lead to different dispersion Eqs. ((14), (9)) for surface and body waves, are usually a possible means for distinction. However, this does not work for leaky waves, because as indicated earlier, the dispersion equation does not depend on the choice of the wave functions as it does for the ordinary slow modes. There is no way to separate the modes except by the sign of the dispersion. The phase velocities ω/k of the slow surface and body waves are decreasing and increasing functions of wavenumber k, respectively (see, for example, Figs. 3, 4 in Edwin & Roberts 1983). This property is used to call these waves of negative and positive dispersion, respectively. A detailed analysis of the dispersion is carried out in the next section. It turns out that the first root of the dispersion Eq. (26) (or (28)) for the leaky slow waves describes the mode of positive dispersion. For that reason it is defined as a slow leaky surface mode. Particular attention has been given to the classification of leaky modes, because the sign of the dispersion is crucial for the properties of nonlinear waves (Sect. 4).

2.5. Shortcomings of zero-order thin-flux-tube approximation

Thus the thin-flux-tube limit exists for all ranges of magnetic fields and sound velocities outside and inside tube. It shows up as the limit of either the slow surface or leaky surface or body waves depending on the value of the external magnetic field. It is remarkable that all of these modes exist only when the effect of the external medium is taken into account. The tube waves in an infinitely thin tube exist because of the effect of the external plasma. A paradox of the thin-flux-tube approximation is that the set of Eqs. ((1)–(5)) of the thin-flux-tube approximation does not depend on parameters of the external plasma. It becomes possible due the absence of the wave dispersion as well as the wave attenuation due to wave emission in the thin tube limit. The ignorance of this hidden problem can lead to erroneous results when the thin-flux-tube approximation is applied to waves in thin tubes with a finite diameter. Slow waves could undergo essential attenuation in thin tubes immersed in a plasma with a "intermediate" magnetic field. This effect is completely lost in the thin-flux-tube approximation.

Intuitively it might be anticipated that the neglect of wave dispersion in the thin-tube approximation is less severe than the neglect of attenuation, because dispersion is weak in thin tubes. However, this is not correct. The neglect of wave dispersion is essential for the nonlinear waves. The effects of dispersion on nonlinear waves in a thin tube are considered in Sect. 5.

3. Dispersion of slow surface and body waves in thin tubes

Once the thin tube limit for an arbitrary external medium is known, it is simple to obtain the approximate dispersion relation for a thin tube with a non-zero diameter. The approximate dispersion relation defines the dispersion, which is not significant for linear waves in a sufficiently thin tube, but which is crucial for the properties of nonlinear waves. The attenuation of waves due to wave emission by the tube can also be estimated from the approximate dispersion equation and is essential for both linear and nonlinear waves.

3.1. Surface waves

For a "weak" external magnetic field, when the condition (18) is valid, the dispersion Eq. (14) for slow surface waves has a single root, which in the thin tube limit tends to zero $j \ll 1$ and equals, approximately, to

$$j^2 \approx -\frac{2\beta_{\rm e} m_{\rm s} x}{\beta\Delta((1+\beta)\delta-\beta_{\rm e})}\frac{K_1(m_{\rm s}x)}{K_0(m_{\rm s}x)}\cdot \tag{30}$$

To obtain the approximate solution of the dispersion equation the modified Bessel functions in (14) are replaced by their asymptotic values $I_0(j) \approx 1$ and $I_1(j) \approx j$ in the limit $j \rightarrow 0$.

For weak dispersion the dimensionless frequency of the surface waves (16) equals approximately

$$\Omega_{\rm s}^2 \approx 1 + \frac{\beta x^2}{(1+\beta)^2(x^2 - j_1^2)}\cdot \tag{31}$$

After substitution of (30) in (31) and taking into account, that $K_0(m_{\rm s}x) \rightarrow (\ln(m_{\rm s}x))^{-1}$ and $m_{\rm s}K_1(m_{\rm s}x) \rightarrow 1$ for $x \rightarrow 0$, we arrive at the approximate dispersion equation

$$\Omega_{\rm s}^2 \approx 1 + \frac{\beta^2\Delta((1+\beta)\delta-\beta_{\rm e})x^2K_0(mx)}{2(1+\beta)^2\beta_{\rm e}}, \tag{32}$$

where the asymptotic behaviour $m_{\rm s}xK_1(m_{\rm s}x) \rightarrow 1$ for $x \rightarrow \infty$ is used and $m_{\rm s}$ is replaced by

$$m = \sqrt{\frac{[\delta(1+\beta)-1][\delta(1+\beta)-\beta_{\rm e}]}{\delta(1+\beta)[\delta(1+\beta)-(1+\beta_{\rm e})]}}, \tag{33}$$

because in the thin tube limit the phase velocity tends to the tube velocity $C(k) = \omega/k \rightarrow C_{\rm T}$ and, consequently, the dimensionless frequency of the slow mode $\Omega_{\rm s} \rightarrow 1$. For a flux tube immersed in a magnetic-free plasma $\beta_{\rm e}^{-1} = 0$, the approximate dispersion Eq. (32) can be reduced to

$$\Omega_{\rm s}^2 \approx 1 - \frac{\Delta\beta^2x^2K_0(mx)}{2(1+\beta)^2}\cdot \tag{34}$$

The dispersion of the slow surface (32) mode is negative, that is, the phase velocity is an decreasing function of the wavenumber. Thus, high harmonics of the slow surface waves in a thin tube immersed in a "weak" magnetic field are running more slowly than the low harmonics.

For the discussion of nonlinear waves in later sections we need to know the short wavelength limit $x \to \infty$ of the dimensionless frequency (phase velocity) Ω of the surface waves. Using the appropriate asymptotic expressions of the modified Bessel functions for large arguments we can rewrite the dispersion Eq. (14) of the surface waves as

$$\left(1+\beta-\beta\Omega_{\rm s}^2\right)m_{\rm s}+((1+\beta)\delta-\beta_{\rm e}\Omega_{\rm s}^2)\Delta\overline{j}=0, \qquad (35)$$

where $\Omega_{\rm s}$ is expressed in the terms of $\overline{j}=j/x$ as

$$\Omega_{\rm s}^2=\frac{2}{1+\sqrt{1-4\beta a^{-1}}},\ a=(1+\beta)^2(1-\overline{j}^{-2}). \qquad (36)$$

The limiting dispersion Eq. (35) for surface waves does not depend on x as expected. The limit $x\to\infty$ can be reached either by $k\to\infty$ for a fixed value of R_0 or by $R_0\to\infty$ for a fixed k. This means that the phase velocity of the surface waves $C(k)=\Omega_{\rm s}C_{\rm T}$ does not depend on the tube radius in the limit of $k\to\infty$. The phase velocity of the surface waves is less than the tube velocity $C(k)<C_{\rm T}$ or $\Omega_{\rm s}<1$. To prove this statement we recall that the dimensionless frequency $\Omega_{\rm s}$ satisfies the algebraic equation

$$\beta\Omega_{\rm s}^4+(\overline{j}^{-2}-1)(1+\beta)^2(\Omega_{\rm s}^2-1)=0. \qquad (37)$$

For $\Omega_{\rm s}=1$ this equation cannot be satisfied for any values of $\overline{j}$. Consequently, for any choice of parameters $\Omega_{\rm s}<1$ and phase velocity $C(k)<C_{\rm T}$ in the limit $k\to\infty$, because the dispersion of surface waves is negative, as it is shown above. Thus, it is clear that the phase velocity of the surface waves, which is equal to the tube velocity in the thin tube limit, decreases when we increase the wavenumber k and for large wavenumbers $k\to\infty$ it tends to a constant value, below the tube velocity.

3.2. Body waves

The dispersion Eq. (9) for the slow body waves has an infinite number of roots. Only for a "strong" external magnetic, when condition (23) is satisfied, does the first root of (9) tend to zero in the thin tube limit. The root equals approximately to

$$j_1^2\approx\frac{2\beta_{\rm e}m_{\rm s}x}{\beta\Delta((1+\beta)\delta-\beta_{\rm e})}\frac{K_1(m_{\rm s}x)}{K_0(m_{\rm s}x)}\cdot \qquad (38)$$

It differs from (30) only by its sign. After substitution of (38) into the approximate expression for the frequency of the slow body waves (11)

$$\Omega_{\rm b}^2\approx 1-\frac{\beta x^2}{(1+\beta)^2(x^2+j_1^2)}, \qquad (39)$$

the approximate dispersion equation for slow body waves is obtained. It is exactly the same as for surface waves (34). But the dispersion for slow body waves differs in sign from that of the surface waves, because the dispersion is proportional to the factor $\delta(1+\beta)-1$, which is negative, when the condition of a "strong" external magnetic field (23) is valid. The factor is positive, when the condition of a "weak" external magnetic field (21) is valid. Thus, high harmonics of slow body waves in a thin tube immersed in a "strong" magnetic field are running faster than the low harmonics.

When we use the asymptotics of the modified Bessel function $K_0(mx)$, we can further simplify the approximate dispersion Eq. (34) for surface waves in a "weak" magnetic field and for body waves in a "strong" magnetic field to

$$\Omega_{\rm s,b}^2\approx 1+\frac{\beta^2\Delta((1+\beta)\delta-\beta_{\rm e})x^2}{4(1+\beta)^2\beta_{\rm e}}\cdot \qquad (40)$$

In the special case of the solar corona, when $\beta_{\rm e}\ll 1$ the approximate dispersion relation (40) for body waves can be further simplified to

$$\Omega_{\rm b}^2\approx 1+\frac{\beta x^2}{4(1+\beta)}\cdot \qquad (41)$$

The first root of the exact dispersion Eq. (9) varies from $j_1=0$ to $j_1=2.41$ while the dimensionless wavenumber increases from $x=0$ to $x=\infty$, because the Bessel function $J_0(j)$ changes its sign for $j=2.405$ and Eq. (9) has not any real roots.

The next roots, as it was mentioned in Sect. 2.3, are in the ranges $j_{1,n-1}<j_n<j_{0,n+1}$. The roots of Eq. (9) exist only for the ranges of j, where the Bessel functions $J_0(j)$ and $J_1(j)$ have the same signs. The approximate dispersion relations for the next roots of the exact dispersion Eq. (9) for body waves in the thin tube limit reads

$$\Omega_{\rm b}^2=1+\frac{\beta x^2}{(1+\beta)^2 j_{1,n}}, \qquad (42)$$

where $j_{1,n}$ are roots of the equation $J_1(j)=0$. This formula is valid for the body waves in a thin tube immersed in a "weak" magnetic field. Thus, the dispersion Eq. (9) for body waves has one more root for a "strong" field than for a "weak" field. This extra root corresponds to the wave mode, which tends to the tube waves in the thin tube limit.

The short wavelength limit $x\to\infty$ of the dimensionles frequency (11) is the same for all slow body modes. It does not depend on the parameters of external plasma

$$\Omega_{\rm b}=\sqrt{1+\beta},\quad C(k)=C_{\rm S}. \qquad (43)$$

Thus, the phase speed of the slow body waves in a thin tube immersed in a "strong" magnetic field varies from the tube speed $C_{\rm T}$ to the sound speed as the wavenumber k increases from zero to infinity.

3.3. Leaky surface waves

When the condition of an "intermediate" external field (23) is satisfied, the approximate dispersion equation can be obtained by a similar procedure to either the dispersion Eq. (26) for the leaky surface waves or the dispersion Eq. (28) for the leaky body waves (28). The result is the same in both cases. The first root of the dispersion Eq. (26) for leaky surface waves for $j \ll 1$ equals approximately

$$j_1^2 \approx \frac{2\beta_e n_s x}{\beta\Delta((1+\beta)\delta - \beta_e)} \frac{H_1^{(2)}(n_s x)}{H_0^{(2)}(n_s x)}. \tag{44}$$

After substitution of (44) into the approximate expression for the frequency of the slow surface mode (31), and taking the limit $x \to 0$ for n_s the approximate dispersion relation for leaky surface waves is obtained

$$\Omega_s^2 \approx 1 + \frac{\imath\beta^2\Delta((1+\beta)\delta - \beta_e)x^2 H_0^{(2)}(nx)}{\pi(1+\beta)^2\beta_e}, \tag{45}$$

where

$$n = \sqrt{\frac{[1-\delta(1+\beta)][\delta(1+\beta)-\beta_e]}{\delta(1+\beta)[\delta(1+\beta)-(1+\beta_e)]}}. \tag{46}$$

The dispersion relation is complex because of the wave emission by the tube. The real part of the dispersion is proportional to $Re(\imath H_0^{(2)}(nx)) = Y_0(nx)$ and, consequently, the dispersion of the leaky slow waves has the same sign as that for slow surface waves. This means that the first root of the dispersion Eq. (26) for leaky surface waves, which tends to zero in the thin tube limit, corresponds truly to leaky surface waves. Other roots of (26) correspond to waves with positive dispersion, and have to be called leaky body waves.

Particular attention has been given to the dispersion of waves because the difference of the sign of the dispersion of slow waves in thin tubes immersed in "weak" and "strong" external magnetic fields is crucial for the properties of nonlinear waves, which are considered below.

4. Second order thin-flux-tube approximation

Zhugzhda (1996) derived a second order thin-flux-tube approximation, which takes into account wave dispersion, as distinct from the zero order approximation discussed above.

The basic set of equations of the second-order approximations reads

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial z} = 0, \tag{47}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B}\right) + \frac{\partial}{\partial z}\left(v\frac{\rho}{B}\right) = 0, \tag{48}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)\frac{p}{\rho^\gamma} = 0, \tag{49}$$

$$\frac{\partial A}{\partial t} + v\frac{\partial A}{\partial z} - 2Au = 0, \tag{50}$$

$$\frac{\partial B}{\partial t} + v\frac{\partial B}{\partial z} + 2Bu = 0, \tag{51}$$

$$p + \frac{B^2}{8\pi} - \frac{A}{2\pi}\left[\rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial z} + u^2\right) + \frac{1}{4\pi}\left(\frac{B}{2}\frac{\partial^2 B}{\partial z^2} - \frac{1}{4}\left(\frac{\partial B}{\partial z}\right)^2\right)\right] = p_{\rm ext}, \tag{52}$$

where $A = \pi R^2$ is the cross section of the tube, R is the radius of the tube and the product vR is the velocity of the tube boundary.

In this approximation the dispersion relation reads in our dimensionless variables

$$\frac{\beta x^2\Omega^4}{4(1+\beta)^2} - \left(1 + \frac{x^2}{4}\right)(\Omega^2 - 1) = 0. \tag{53}$$

For weakly dispersive slow waves it can be reduced to

$$\Omega^2 \approx 1 + \frac{\beta x^2}{4(1+\beta)}. \tag{54}$$

This equation coincides exactly with the approximate dispersion Eq. (41), which was obtained from the exact dispersion Eq. (9) for a "strong" field with the additional assumption that $\beta_e \ll 1$. This makes it clear, that the second order approximation derived by Zhugzhda (1996) works correctly for a "very strong" field, when the condition (23) is satisfied along with $\beta \sim \beta_e \ll 1$. It comes as no surprise, that the second order approximation works well for a strong magnetic field, because the condition of a fixed external pressure $p_{\rm ext} = const.$ is used in the thin tube approximation.

Moreover, the dispersion Eq. (53) provides the exact value of the short wavelength limit $x \to \infty$ for the dimensionless frequency $\Omega = 1+\beta$ (to compare with (43)). Thus, the second order thin-flux-tube approximation works also for a thick tube, when $x = kR_0 \gg 1$. It seems likely, that the second order thin flux tube approximation (Zhugzhda 1996) is a two mode approximation, which describes the first slow and fast modes in the tube (Zhugzhda 2001).

The second order thin tube approximation needs a detailed exploration similar to the current analysis of the zero order thin-flux-tube approximation. However, this is beyond the scope of our present paper. But it is evident, that there is a basic difference between the zero and the second order approximations. The last one is a powerful tool for exploring nonlinear waves by analytical and numerical methods, because it takes into account the wave dispersion.

5. Weakly nonlinear waves in thin tubes

Within the thin-flux-tube approximation the waves of finite amplitude cannot but form shocks. This is not the case for a thin tube of non-zero diameter due to the effect of the wave dispersion. When the wave amplitudes are sufficiently small for the nonlinear effects to be compensated by the dispersion, shock fronts do not appear and the waves are weakly nonlinear. The theory of weakly nonlinear waves in thin tubes embedded in an non-magnetic plasma was developed by Roberts (1985). The above analysis of the thin tube limit makes it possible to extend the theory of Roberts to an arbitrary external medium. We follow mainly the approach developed by Zhugzhda (2000).

5.1. Generalized Leibovich-Roberts equation (GLR)

Roberts (1985) derived the equation for weakly nonlinear surface slow waves in a thin flux tube embedded in a magnetic field free plasma. The equation is known as the Leibovich-Roberts (LR) equation and reads

$$\frac{\partial v}{\partial t}+C_{\mathrm{T}}\frac{\partial v}{\partial z}+bv\frac{\partial v}{\partial z}+\Delta_{\mathrm{T}}\frac{\partial^3}{\partial z^3}\int\limits_{-\infty}^{+\infty}\frac{v(z',t)\mathrm{d}z'}{[\lambda^2+(z'-z)^2]^{1/2}}=0,$$

where (55)

$$b=\frac{C_{\mathrm{A}}^2[3C_{\mathrm{S}}^2+(\gamma+1)C_{\mathrm{A}}^2]}{2(C_{\mathrm{S}}^2+C_{\mathrm{A}}^2)^2},\ \Delta_{\mathrm{T}}=\frac{1}{8}\frac{\rho_{\mathrm{e}}}{\rho_0}\left(\frac{C_{\mathrm{T}}}{C_{\mathrm{A}}}\right)^4C_{\mathrm{T}}k^2R_0^2,$$

$$\lambda=C_{\mathrm{T}}R_0/C_{\mathrm{A}}.$$

The nonlinear term in this equation was obtained by using the thin-flux-tube approximation. This means that nonlinear effects have only been taken into account inside the tube. The dispersion term was obtained by looking at the effect of the surrounding plasma, because it is the surrounding plasma that causes dispersion of surface slow waves.

The LR equation can be derived by the method of Whitham (1974), when the dispersion law is known. Whitham showed, that the 1-D wave equation for a given dependence of the phase velocity $C(k)$ on the wave number reads

$$\frac{\partial v(z,t)}{\partial t}-\int\limits_{-\infty}^{+\infty}K(z'-z)\frac{\partial v(z',t)}{\partial z'}\mathrm{d}z'=0, \tag{56}$$

$$K(z)=\int\limits_{-\infty}^{+\infty}C(k)\exp(-ikz)\mathrm{d}k.$$

We can rewrite the approximate dispersion Eq. (32) for slow surface waves, when the condition (19) $C_{\mathrm{T}}^2>C_{\mathrm{Ae}}^2$ is satisfied, and for slow body waves, when the condition (22) is valid $C_{\mathrm{T}}^2<C_{\mathrm{Te}}^2$, as

$$C(k)\approx C_{\mathrm{T}}-\frac{1}{4}\left(\frac{C_{\mathrm{T}}}{C_{\mathrm{A}}}\right)^4\left(1-\frac{C_{\mathrm{Ae}}^2}{C_{\mathrm{T}}^2}\right)\frac{\rho_{0\mathrm{e}}}{\rho_0}C_{\mathrm{T}}k^2R_0^2K_0(mx). \tag{57}$$

The dispersion for surface waves is negative since then the factor $(1-C_{\mathrm{Ae}}^2/C_{\mathrm{T}}^2)>0$ is positive, when condition (19) holds. For body waves the dispersion is positive, since under the condition (22) the factor $(1-C_{\mathrm{Ae}}^2/C_{\mathrm{T}}^2)>0$ is positive because of the inequality $C_{\mathrm{Te}}^2<C_{\mathrm{Ae}}^2$. For the dispersion (57) the one-dimensional wave Eq. (56) reads

$$\frac{\partial v}{\partial t}+C_{\mathrm{T}}\frac{\partial v}{\partial z}+\Delta_{\mathrm{T}}\int\limits_{-\infty}^{+\infty}\frac{\partial^3 v(z',t)}{\partial z'^3}\frac{\mathrm{d}z'}{[\lambda^2+(z'-z)^2]^{1/2}}=0, \tag{58}$$

where the integral transformation of the modified Bessel function K_0 (Bateman & Erdèlyi 1954) is used and,

$$\Delta_{\mathrm{T}}=\frac{1}{8}\left(\frac{C_{\mathrm{T}}}{C_{\mathrm{A}}}\right)^4\left(1-\frac{C_{\mathrm{Ae}}^2}{C_{\mathrm{T}}^2}\right)\frac{\rho_{0\mathrm{e}}}{\rho_0}C_{\mathrm{T}}k^2R_0^2, \tag{59}$$

$$\lambda^2=m^2R_0^2=\frac{(C_{\mathrm{Se}}^2-C_{\mathrm{T}}^2)(C_{\mathrm{Ae}}^2-C_{\mathrm{T}}^2)}{(C_{\mathrm{Ae}}^2+C_{\mathrm{Se}}^2)(C_{\mathrm{Te}}^2-C_{\mathrm{T}}^2)}R_0^2. \tag{60}$$

It is straightforward to verify by substitution of $v(z,t)=V\exp(\imath(\omega t-kz))$ into (58), that this wave equation corresponds to the dispersion equation. The same dispersion equation appears, when the order of the integration and differentiation is reversed, as it is to be done for the LR equation. To obtain the GLR equation for a thin tube embedded in a magnetic plasma the nonlinear term from the LR Eq. (55) has to be added to the wave Eq. (58)

$$\frac{\partial v}{\partial t}+C_{\mathrm{T}}\frac{\partial v}{\partial z}+bv\frac{\partial v}{\partial z}+\Delta_{\mathrm{T}}\frac{\partial^3}{\partial z^3}\int\limits_{-\infty}^{\infty}\frac{v(z',t)\mathrm{d}z'}{\sqrt{\lambda^2+(z'-z)^2}}=0. \tag{61}$$

The GLR equation has to be reduced to the LR Eq. (55), when the plasma outside the tube is free of a magnetic field. In fact the parameter Δ_{T} coincides exactly with the one in (55) obtained by Roberts (1985). The parameter λ is reduced to

$$\lambda^2=\left(1-\frac{C_{\mathrm{T}}^2}{C_{\mathrm{Se}}^2}\right)R_0^2, \tag{62}$$

and this parameter is reduced to (55) only in the special case of $C_{\mathrm{S}}=C_{\mathrm{Se}}$, which was considered by Roberts (1985). Thus, the GLR equation derived for weakly nonlinear slow waves in a thin flux tube immersed in magnetic plasma is more general than the GL equation, which is only valid, when $C_{\mathrm{Ae}}=0$ and $C_{\mathrm{S}}=C_{\mathrm{Se}}$. However, the Eq. (61) is not valid for leaky surface waves, when there is emission of waves by the tube. The GLR Eq. (61) is only valid for $\lambda^2>0$, because the integral transformation, which was used to obtain this equation, works only for this case (Bateman & Erdèlyi 1954).

In order to obtain the nonlinear equation for the leaky surface waves we have to extend the method of Whitham (1974) to $\lambda^2<0$, which corresponds to the condition (24) for the leaky surface waves to exist in a thin tube. We can write the approximate dispersion Eq. (45) for this case as

$$C\approx C_{\mathrm{T}}-\frac{\imath}{2\pi}\left(\frac{C_{\mathrm{T}}}{C_{\mathrm{A}}}\right)^4\left(1-\frac{C_{\mathrm{Ae}}^2}{C_{\mathrm{T}}^2}\right)\frac{\rho_{0\mathrm{e}}}{\rho_0}k^2C_{\mathrm{T}}R_0^2H_0^{(2)}(nx). \tag{63}$$

The imaginary part of the phase velocity describes wave damping due to the wave radiation by the tube. The 1-D nonlinear equation for waves with dispersion and damping of this type reads

$$\frac{\partial v}{\partial t}+C_{\mathrm{T}}\frac{\partial v}{\partial z}+bv\frac{\partial v}{\partial z}+$$

$$+\Delta_{\mathrm{T}}\frac{\partial^3}{\partial z^3}\left[\int\limits_{|\lambda|+z}^{\infty}+\int\limits_{-\infty}^{-|\lambda|+z}\frac{v(z',t)\mathrm{d}z'}{\sqrt{(z'-z)^2+\lambda^2}}\right]=0, \tag{64}$$

where Δ and λ are defined by (59) and (60) and, the integral transformation of the Hunkel function $H_0^{(2)}$ (Bateman & Erdèlyi 1954) is used. In this case $\lambda^2 < 0$ due to the condition (24), while $\Delta_{\rm T} < 0$ and the dispersion is positive. When $C_{\rm T}^2 \to C_{\rm Ae}^2$, $\lambda \to 0$ the GLR Eq. (64) is transformed into Leibovich equation

$$\frac{\partial v}{\partial t} + C_{\rm T}\frac{\partial v}{\partial z} + bv\frac{\partial v}{\partial z} + \Delta_{\rm T}\frac{\partial^3}{\partial z^3}\int\limits_{-\infty}^{+\infty}\frac{v(z',t){\rm d}z'}{|z'-z|} = 0, \qquad (65)$$

which has been derived by Leibovich (1970) for waves on a cylindrical vortex core. When $C_{\rm T}^2 \to C_{\rm Te}^2$, the dispersion of leaky slow surface waves tends to zero and the generalized GLR equation is reduced to a nonlinear wave equation without dispersion. The last case corresponds to the transition from surface to body waves, when the sign of the dispersion is reversed. Thus, the GLR Eqs. ((61), (64)) cover all range of parameters of the plasma outside the tube.

5.2. Hidden problems of GLR and LR equation

On first sight the derivation of GLR equation gives a solid basis for studying weakly nonlinear waves in a thin tube and the only problem to be handled is this nonlinear equation. But there are hidden problems in both the LR and the GLR equation. An analytical solution of the Leibovich equation (Leibovich 1970) is not yet known for waves on a cylindrical vortex core. Solitary solutions have been found experimentally by Pritchard (1970) and numerically by Leibovich & Randall (1972). As for the LR equation, Bogdan & Lerche (1988) claimed that there are soliton-like solutions of this equation. Numerical solutions of the LR equation have been obtained by Weisshaar (1989). He pointed out that the scale parameter λ, which appears in the LR equation but not in the Leibovich equation, introduces essential differences in the properties of the solutions. The numerical analysis (Weisshaar 1989) showed, that the solitary wave solutions of the LR equation are not self-similar and exist only up to some critical amplitude. The critical amplitude shows up because phase velocity as defined by (57) has a maximum. If the amplitude exceeds a critical value, the dispersion appears to be insufficient to smooth out the wave front and a shock has to form. The same effect has to be present for the GLR equation, because the approximate dispersion Eq. (57) is valid for both a non-magnetic and a magnetic plasma outside the tube. Thus, the GLR equation describes the behaviour of weakly nonlinear waves, whose phase velocity has an extremum in the linear limit. In the limit of large and small wavenumbers the phase velocity of the linear waves is the same.

As a matter of fact the derivation of the GLR equation suffers from an inconsistency as the equation takes large wavenumbers into account, while its derivation is based on the approximate dispersion relation for small wavenumbers, which has an extremum. It is shown earlier in this paper, that the phase speed of surface waves decreases and tends to a velocity below the tube velocity $C_{\rm T}$, without having a minimum. While the phase velocity of body waves for a "strong" external field increases and tends to the sound speed without attaining a maximum. A similar problem exists for the leaky surface waves, because the dispersion and absorption terms in the approximate dispersion Eq. (63) become oscillatory for large k. The numerical solutions of the exact dispersion Eq. (26) reveal that the dispersion and absorption of leaky surface waves depend monotonically on the wavenumber, which is the only physically admissible behaviour. Therefore, the GLR and LR equations are, strictly speaking, not suited for treating weakly nonlinear body and surface waves with amplitudes, which are affected by the existence of an extremum in the phase velocity.

5.3. Korteweg-de Vries equation (KdV)

Zhugzhda & Nakariakov (1997a) derived the KdV equation for slow body waves in the thin flux tube. They used the second order thin-flux-tube approximation (Zhugzhda 1996). First, they obtained the nonlinear wave equation by the expansion of the equations up to the second order of the amplitude. Second, they obtained the KdV equation from the nonlinear wave equation by multi-scale analysis. It was revealed above, that the second-order thin-flux-tube approximation describes slow body waves in a thin tube immersed in a plasma with a "very strong" magnetic field, that is in the limit $\beta_{\rm e} \to \infty$. Because of the coefficient b in front of the nonlinear term is known, the use of Whitham's equation makes it possible to obtain the KdV equation for both slow surface waves in a thin tube with a "weak" field and for slow body waves in a thin tube with a "strong" field. In the co-moving coordinate system it reads

$$C_{\rm T}\frac{\partial v}{\partial \tau} + bv\frac{\partial v}{\partial z} + \Delta_{\rm T}\frac{\partial^3 v}{\partial z^3} = 0, \qquad (66)$$

where $\tau = t - C_{\rm T}^{-1}z$ is the slow time. Roberts (1985) and Zhugzhda & Nakariakov (1997a) derived the corresponding nonlinear term from the zero order and the second order thin-flux-tube approximations, respectively. The nonlinear term is exactly the same in both derivations. It is no surprise, because the dispersion, which is taken into account by the second-order approximation, does not affect the nonlinear term. The general KdV Eq. (66) is reduced to Zhugzhda & Nakariakov (1997a) equation except for the so called correction coefficient. The introduction of the correction of the second-order approximation was proposed by Zhugzhda (1996) because it was assumed, that it describes slow body mode with dispersion defined by (42) for $n = 1$. We revealed, that the second order thin-tube approximation describes slow body waves with the dispersion defined by (40) in the limit of a "very strong" external field $\beta_{\rm e} \to \infty$. In this case the corrections of the thin-tube approximation and the KdV equation are not needed.

The KdV Eq. (66) is useful for astrophysical applications, because its solutions are simple and well explored. On the other hand, the absence of exact analytical solutions of the LR equation prevents us from making qualitative statements on weakly nonlinear waves in astrophysical plasmas. The KdV equation is valid for any amplitudes of the solitons, as unbounded increasing of the dispersion is assumed. As it was shown above the phase velocity of the slow surface and the body waves tend to a finite limit for $k \to \infty$. Thus, the KdV equation suffices only for describing solitons, which have a relatively small amplitude, and for which the high harmonics $kR_0 \gg 1$ are not essential.

6. Discussion

The paper shows that the thin-flux-tube approximation is valid for any plasma environment of the tube. In fact, the thin flux tube approximation is used to treat the dynamics of thin flux tubes of a small diameter, but not infinitely thin tubes, which are just a very useful mathematical limit. Our analysis identifies which physical effects are dropped, when the thin flux tube approximation is applied to thin tubes with a non-zero diameter. In addition, it makes it possible to take into account the effects that were omitted originally, namely, the wave dispersion and emission. Dispersion, omitted originally in the thin tube approximation, is not very important for linear waves in thin flux tubes, because the discrepancy between the tube velocity and exact phase velocity in a thin tube is so small that the wave propagation along the tube is not changed radically. Attenuation of waves due to radiation of waves by the tube, which occurs when the condition (25) is satisfied, is a different story. In this case linear waves are not trapped in the tube.

The attenuation due to the wave radiation by the tube decreases with decreasing diameter. This is crucial for a scenario of the heating of the upper solar atmosphere by waves excited in tubes by convective turbulence. There are two main scenarios in connection with the effect of wave radiation. The first scenario operates, when the loop in the corona is sufficiently thin and the wave damping due to radiation is a negligibly small. In this case the waves form shocks and dissipate. If the loop is sufficiently thick most of the wave energy goes into radiation of waves, and shocks do not appear in the tube. In this second scenario the dissipation of the radiated waves has to be considered. Thus, the classical scheme of heating of flux tubes in the chromosphere or corona is valid only for rather thin flux tubes. This effect can be considered as a possible explanation of the fine structure of the upper solar atmosphere. In addition, the condition for wave radiation (23) can be rewritten as

$$\delta = \frac{T_{0\mathrm{e}}}{T_0} < \frac{1+\beta_\mathrm{e}}{1+\beta} \approx 1 \qquad (67)$$

in the limit $\beta_\mathrm{e}, \beta \ll 1$, which is valid for the solar corona. Thus, slow waves are attenuated in the tube, when the tube is hotter than its surrounding plasma. Heating of the hot loops in the corona by slow waves is possible only if they are sufficiently thin for attenuation due to radiation to be a negligibly small. At the same time the condition of a "strong" field (21) in the same limit $\beta_\mathrm{e}, \beta \ll 1$ reads

$$\delta = \frac{T_{0\mathrm{e}}}{T_0} > \frac{1+\beta_\mathrm{e}}{1+\beta} \approx 1, \qquad (68)$$

so that cold loops in the corona can be heated by slow waves.

Waves of finite amplitude in an infinitely thin tube governed by the thin-flux-tube approximation form shocks after some time due to the absence of dispersion. Waves in thin tubes of finite diameter however are dispersive. Dispersion can balance the nonlinear effects and smooth out the shock fronts so that weakly nonlinear waves can exist in thin tubes with a non-zero diameter. Moreover the key properties of the weakly nonlinear tube modes in a thin flux tube depend on the value of the magnetic field in the surrounding plasma. The tube mode appears as a running hump of hot dense plasma, when there is not any magnetic field or when the magnetic field is relatively weak outside the tube $C_\mathrm{T}^2 > C_\mathrm{Ae}^2$. If the magnetic field is relatively strong outside the tube $C_\mathrm{T}^2 < C_\mathrm{Te}^2$, the tube mode appears as a running narrowing of the tube, where the plasma is cooled down, the density is decreased and the plasma is accelerated. This happens because of the different signs of the dispersion law. A detailed description of the effect was done by Zhugzhda & Nakariakov (1997a,b). Besides, a breaking of a tube mode can be different for these two limiting cases. Breaking of hot humps of large amplitude has to be connected with formation of shocks on the front side of the disturbances. In the case of a tube narrowing, a breaking can result from the cooling and acceleration of the plasma in the neck of the narrowing (Zhugzhda & Nakariakov 1997b). This is a very different scenario for astrophysical plasmas. In the case of the solar corona humps can occur in the hot loops, while narrowings in the cold loops. It has been found, that the second-order thin-flux-tube approximation (Zhugzhda 1996) is ideally suited for a simulation of slow waves in the case $\beta \ll 1$, because it provides the correct description of the dispersion and includes all nonlinear effects. Thus, simulations have to show solitons for relatively small amplitudes of the disturbances, and its breaking with shock front formation with increasing of the soliton amplitudes.

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