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The effect of time-dependent random mass density field on frequencies of solar sound waves

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Abstract. The effect of a time-dependent random mass density field on the frequencies and amplitudes of solar p-modes approximated as sound waves is considered by analytical perturbative means and numerical simulations for one-dimensional hydrodynamic equations. The analytical results, which are worked out for a Gaussian spectrum of the random mass fluctuations, show frequency increase and amplitude amplification, in agreement with numerical simulations.

Key words. convection – Sun: oscillations – turbulence

1. Introduction

Our present investigation is mainly motivated by the long standing riddle about discrepancies existing between the measured and computed frequencies of the solar p-modes (e.g. Christensen-Dalsgaard 1998). We investigate the contribution to these discrepancies by density fluctuations in the turbulent convective layer of the Sun where p-modes penetrate.

This issue attracted interest in the past. For instance, Lou & Rosner (1986) and Li & Zweibel (1987) considered the decay of an Alfvén wave which propagated through a medium having time-dependent random density fluctuations. More recently, Nocera et al. (2001) showed that p-modes propagating in a static medium having a random density with a Gaussian correlation function are accelerated and damped.

A variety of behaviours of waves propagating in random media emerges also in Seismology and Oceanography. For instance Kawahara (1976) showed that a time-dependent random field associated with bottom inhomogeneities leads to amplitude attenuation and increase (decrease) of low (high) frequency self-modulated surface gravity waves. Benilov & Pelinovsky (1989) provided examples of time-dependent random media whose high (low) frequency fluctuations lead to wave amplification (damping). Muzychuk (1975) pointed out that space- and time-dependent fluctuations lead to a reduction of the mean field damping and eventually to enhancement of this field.

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Numerical simulations of sound wave propagation in time-dependent random media were performed by Juvé et al. (1999) whose approach was based on the Helmholtz equation (e.g. Sobczyk 1985). Shapiro & Hubral (1995) found acceleration and attenuation of elastic waves propagating in one-dimensional media having a space-dependent random mass density.

The above-mentioned results show that random fields affect both the wave amplitudes and frequencies in an apparently contradictory way. In view of the wide range of possible effects revealed by both observation and theory, a question arises as to whether those effects are due to diverse physical states of randomness and may therefore be conducive to any insight into the properties of turbulence.

In other words, the final outcome of the interaction of an initially coherent wave with a random background is sensitive to the properties of turbulence. Put in this way, the study of turbulence driven frequency shifts of solar global oscillations acquires a powerful diagnostic allure. The value of such a study can hardly be overestimated, as it deals with a layer of the solar atmosphere well beyond the possibility of direct optical observation.

In focusing on the effects of turbulence it is convenient to keep the physics of p-mode propagation as simple as possible. Fortunately, without much loss of realism, this is possible for large wavenumber modes, which nowadays can be observed up to a spherical degree of a few thousands and which basically reduce to plane sound waves (Swisdak & Zweibel 1999). This simplification allows us to elaborate on the physics of the random density fluctuations. As in the work by Nocera et al. (2001), any further assumptions concerning random fluctuations are verified numerically: thus we calculate the frequencies of p-modes by both analytical and numerical means and we show that both results are in good agreement.

As an intermediate step towards the study of the fully space- and time-dependent randomness, in the present work we analyze the influence on the frequencies and amplitudes of sound waves of a medium whose mass density randomly varies in time only. In so doing we envisage the opposite limit of the static situation considered by Nocera et al. (2001). The major result of our investigation will be that, for a Gaussian correlation function of the random mass fluctuations, p-modes are *accelerated* and *unstable*. This supports the conjecture that apparently contradicting results may be found, depending on the spectral properties of the random field.

The outline of our work is as follows. In Sect. 2 we work out the dispersion relation for p-modes propagating in a medium whose density fluctuates randomly in space and time. The solution of the dispersion relation in the case of fluctuations which vary in time only is calculated analytically in Sect. 3. The frequencies coming from a direct numerical simulation of a model equation for p-modes are found in Sect. 4. Conclusions and comparison with the work of other authors are drawn in Sect. 5.

2. Hydrodynamic equations

When the order of p-modes is large and their wavelength is small in comparison to the scale of spatial inhomogeneity, they may be approximated as plane sound waves (Swisdak & Zweibel 1999) governed by hydrodynamic equations:

$$\varrho_t + (\varrho V)_x = S_\varrho, \quad (1)$$

$$\varrho(V_t + VV_x) = -p_x, \quad (2)$$

$$p_t + (pV)_x = (1-\gamma)pV_x. \quad (3)$$

Here ϱ is the mass density, V is the x-component of the velocity vector and p is the pressure; the indices x and t denote the spatial and temporal partial derivatives, e.g. $V_x \equiv \partial V/\partial x$; S_ϱ represents the external mass flux which is described in detail in Sect. 4. Gough (1994) showed that such approximation holds even for a moderate order of the p-mode.

Now observations indicate that the solar granules present in the convection zone, where the p-modes penetrate, are in a turbulent state which can drive random mass density fluctuations (e.g. Stein & Nordlund 1998). The influence of such a turbulent state was analyzed by Brüggen (2000) and Bi & Xu (2000) whose analytical linewidths for the p_1-mode fits a part of the observational data (Duvall et al. 1998). To describe such a turbulent state we assume that the equilibrium mass density can be written as follows:

$$\varrho_e(x,t) = \varrho_0 + \varrho_r(x,t), \quad (4)$$

where $\varrho_0 =$ const. and ϱ_r is a random function such that its statistical ensemble average (e.g. Sobczyk 1985) is zero: $\langle\varrho_r\rangle = 0$.

In the limit of small amplitude waves we can linearize Eqs. (1)–(3) to obtain

$$V_{tt} - c_e^2 V_{xx} + [\varrho_{et}/\varrho_e]V_t = 0,\ c_e = [\gamma p_0/\varrho_e]^{1/2}, \quad (5)$$

where $\varrho_e(x,t)$ and $p_0 =$ const. correspond to the equilibrium mass density and pressure, respectively.

As a consequence of the presence of a random field we introduce the following expansion:

$$V(x,t) = \langle V(x,t)\rangle + V'(x,t), \quad \langle V'\rangle = 0. \quad (6)$$

Here $\langle V\rangle$ and V' have the meaning of the coherent and random fields (e.g. Ostashev 1994), respectively. Substituting this expansion into Eq. (5) and using a *weak* random field approximation (e.g. Howe 1971), we obtain the random dispersion relation

$$\omega^2 - c_0^2k^2 = \omega^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\omega'^2 E(k'-k, \omega'-\omega)\mathrm{d}k'\mathrm{d}\omega'}{\omega'^2 - c_0^2k'^2}, \quad (7)$$

where $c_0 = [\gamma p_0/\varrho_0]^{1/2}$ is the sound speed in the deterministic medium and $E(k,\omega)$ is the Fourier transform of the correlation function defined as

$$E(k,\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(kx-\omega t)} R(x,t)\mathrm{d}x\mathrm{d}t,$$
$$R(|x_2 - x_1|, |t_2 - t_1|) = \langle \varrho(x_2,t_2)\varrho(x_1,t_1)/\varrho_0^2\rangle. \quad (8)$$

3. Time-dependent random mass density

In analyzing Eq. (7), several approximations can be made: Nocera et al. (2001) considered the limit of time-independent random fluctuations; here we restrict our analysis to the opposite limit of spatially homogeneous, but time-dependent random density fluctuations: $\varrho_r = \varrho_r(t)$. We introduce the dimensionless variables

$$K = kc_0l_t, \qquad \Omega = \omega l_t, \quad (9)$$

where l_t is the correlation time. Then, dispersion relation (7) attains the following form:

$$\Omega^2 - K^2 = \Omega^2 \int_{-\infty}^{+\infty} \frac{\Omega'^2 E(\Omega'-\Omega)\mathrm{d}\Omega'}{\Omega'^2 - K^2}. \quad (10)$$

Now, we introduce the Gaussian spectrum

$$E(\Omega) = \frac{\sigma^2}{\pi}\mathrm{e}^{-\Omega^2}, \quad (11)$$

where σ is the variance. For this spectrum, Eq. (10) simplifies to the following form:

$$\Omega^2 - K^2 = \frac{\sigma^2}{\pi}\Omega^2 \int_{-\infty}^{+\infty} \frac{\Omega'^2 \mathrm{e}^{-(\Omega'-\Omega)^2}}{\Omega'^2 - K^2}\mathrm{d}\Omega'. \quad (12)$$

Since the density fluctuations are assumed to be "small" the quantity σ in Eq. (11) is a small number and the solution of Eq. (12) can be expanded in terms of σ^2 (Nocera et al. 2001):

$$\Omega = K + \sigma^2\Omega_2 + \cdots. \quad (13)$$

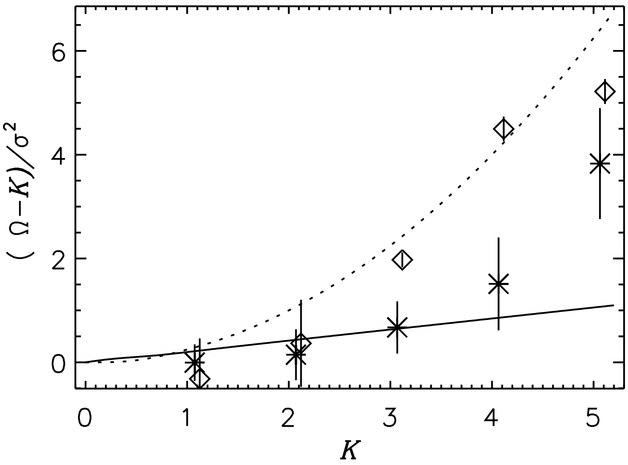

Fig. 1. The real (solid line) and imaginary (dashed line) parts of $(\Omega - K)/\sigma^2$ as functions of the wavenumber K for the dispersion relation of Eq. (17). A time-dependent random mass density field leads to frequency increase and amplification of sound waves. Numerical results for $V_0 = 10^{-2}\ c_0$ and $\sigma = 0.1$ are denoted by stars (real part) and diamonds (imaginary part). The error bars denote standard deviations.

Substituting this expression into Eq. (12), replacing Ω by K on the right hand side of the dispersion relation, we obtain

$$2\pi\Omega_2 = \sqrt{\pi}K\left[1 + \frac{K}{2}(Z(0) - Z(-2K))\right], \tag{14}$$

where

$$Z(a) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\xi^2}\mathrm{d}\xi}{\xi - a} \tag{15}$$

is the plasma dispersion function (Fried & Conte 1961). Using the expression for the plasma dispersion function for a real argument,

$$Z(x + i0) = i\sqrt{\pi}\mathrm{e}^{-x^2} - 2D(x), \tag{16}$$

we obtain

$$2\pi\Omega_2 = \sqrt{\pi}K\left[1 - KD(2K)\right] + i\frac{\pi}{2}K^2\left(1 + \mathrm{e}^{-4K^2}\right), \tag{17}$$

where $D(x) = \mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\mathrm{d}t$ is the Dawson integral (Press et al. 1992).

Figure 1 presents the dispersive curves which follow from Eq. (17). The solid (dashed) line shows the real (imaginary) part of Ω_2 as a function of the wave number K. Since $\Re\Omega_2 > 0$, we conclude that a time-dependent random density field leads to a frequency increase with respect to the case of a deterministic medium: this increase is higher for higher values of K. A most striking result is that, while the solution of the dispersion relation for a time-independent random field revealed wave attenuation (Nocera et al. 2001), the present study shows that the sound waves are amplified by a time-dependent random field ($\Im\Omega_2 > 0$ in Fig. 1). An interpretation of this result will be given in Sect. 5. From a physical point of view, wave amplification takes place at the expenses of the random energy: a higher amplification occurs for higher values of K, i.e. for shorter waves.

4. Numerical simulations for hydrodynamic equations

In this section, we present the results of the numerical simulations for Eqs. (1)–(3). These simulations are performed with a use of the CLAWPACK code (LeVeque 1997), which is a packet of Fortran routines for solving hyperbolic equations. The code utilizes the Godunov-type method (e.g. Murawski & Tanaka 1997, and references therein). Initially, at $t = 0$, the equilibrium state is set as follows:

$$\varrho_{\mathrm{e}} = \varrho_0 = \text{const.}, \quad p = p_0 = \text{const.} \tag{18}$$

Waves are excited impulsively through the initial condition

$$V(x, t = 0) = V_0\ \mathrm{e}^{-(x - x_0)^2}, \tag{19}$$

where V_0 is the amplitude of the initial pulse and $x_0 = 41c_0l_{\mathrm{t}}$ is its position. Periodic boundary conditions are applied at the edges of the simulation region, $x = 0$ and $x \simeq 82c_0l_{\mathrm{t}}$.

The random field is seeded through the term S_ϱ in Eq. (1) chosen so that (Juvé et al. 1999)

$$S_\varrho = \frac{\partial\varrho_{\mathrm{r}}}{\partial t}, \ \varrho_{\mathrm{r}}(t_m) = \sqrt{\frac{2}{N}}\Re\sum_{n=0}^{N-1}\bar{\varrho}(\Omega_n)\mathrm{e}^{(-2\mathrm{i}\pi m\frac{n}{N} + i\phi_n)}. \tag{20}$$

Here time and frequencies are sampled over $N \simeq$ 8200 points

$$0 \le t_n = n\Delta t \le t_{N-1} = 131, \ n = 0, 1, 2, \cdots, N - 1,$$
$$\Omega_n = \frac{2\pi n}{N\Delta t}, \ n = 0, 1, 2, \cdots, N - 1, \ \Delta t = \frac{t_{N-1}}{N};$$

$0 \le \phi_n \le 2\pi$ is a uniformly distributed random phase computed by the random number generator `ran1` (Press et al. 1992). By choosing the amplitude $\bar{\varrho}(\Omega_n)$ as

$$\bar{\varrho}(\Omega_n) = \sqrt{E(\Omega_n)}, \tag{21}$$

Eq. (20) leads to

$$\langle\varrho_{\mathrm{r}}(t_{m_1})\varrho_{\mathrm{r}}(t_{m_2})\rangle = \frac{2}{N}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sqrt{E(\Omega_{n_1})E(\Omega_{n_2})}$$
$$\times\left\langle\Re\left(\mathrm{e}^{(-2\mathrm{i}\pi m_1\frac{n_1}{N} + \mathrm{i}\phi_{n_1})}\right)\Re\left(\mathrm{e}^{(-2\mathrm{i}\pi m_2\frac{n_2}{N} + \mathrm{i}\phi_{n_2})}\right)\right\rangle$$
$$= \frac{1}{N}\Re\sum_{n=0}^{N-1}E(\Omega_n)\mathrm{e}^{[-2\mathrm{i}\pi(m_1 - m_2)\frac{n}{N}]} = R(|t_{m_2} - t_{m_1}|), \tag{22}$$

$R(|t_2 - t_1|)$ being the correlation function of Eq. (8) for space independent fluctuations.

Figure 2 shows the correlation function (top panel) and the random mass density as a function of time for a particular realization of the random density field (bottom

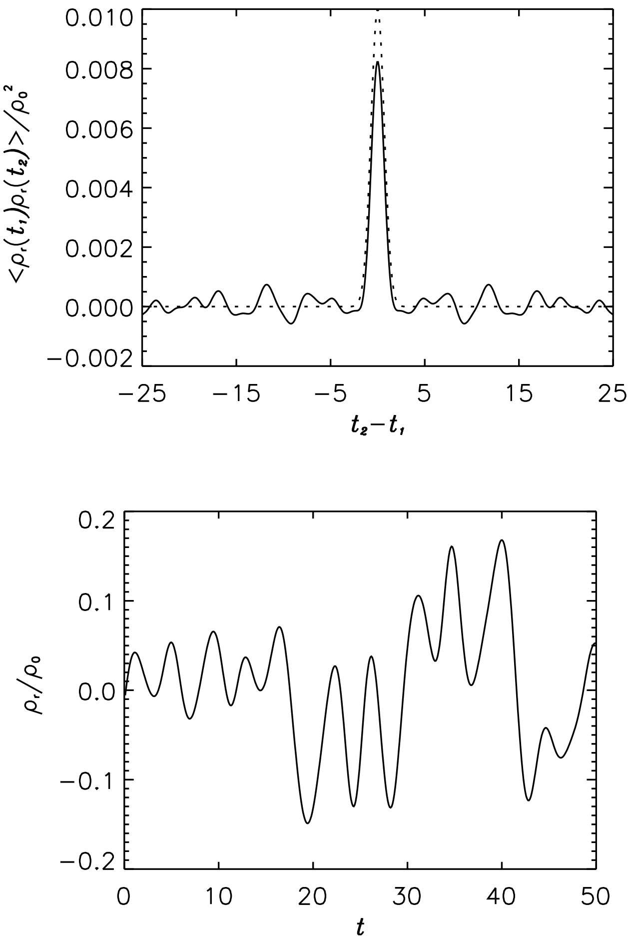

Fig. 2. Top panel: numerical (solid line) and analytical (dashed line) correlation functions for the Gaussian spectrum of Eq. (11). The numerical correlation function is obtained by statistical averaging over 500 realizations of the random field. Bottom panel: random mass density as a function of time t for a typical realization of random medium.

panel). The correlation function is obtained by averaging over 500 realizations. As a consequence of the finite number of realizations the numerical correlation function departs slightly from the analytical Gaussian form.

In Fig. 1 the frequency of maximum spectral concentration of the numerical solution of Eq. (5) is compared, for each wavenumber, with the corresponding analytical value as found in Sect. 3. The span of the error bars equals the standard deviation of the frequency shift over the random density statistical ensemble. We were able to observe frequencies which essentially agree with the analytical prediction. A qualitative difference has been found for long waves ($K \simeq 1$) for which numerical findings reveal wave deceleration and damping while the analytical results show frequency increase and wave amplification. Amongst the several sources of discrepancy between this numerical result and the analytical prediction a most important one is the breakdown of the statistical properties of the random wave operator which are necessary for the derivation of the random dispersion relation in Eq. (10).

5. Summary and discussion

In this paper we presented an analytical and numerical study of the propagation of high wavenumber p-modes (approximated as plane sound waves) in a medium with *time-dependent* random mass density fluctuations. The major result of our investigation is that these modes speed up and become unstable as they propagate; this latter possibility was excluded, on quite general grounds, in the case of a *time-independent* random density field (Howe 1971).

This apparently counter-intuitive instability may be understood by drawing a comparison with randomly perturbed swings: indeed, if $\rho_{\rm r}$ is independent of x, we can Fourier analyze Eq. (5) to get

$$[\rho_{\rm e}^{1/2}v]_{tt} + \left\{[kc_{\rm e}]^2 - ([\ln\rho_{\rm e}]_t)^2/4 - [\ln\rho_{\rm e}]_{tt}/2\right\}\left[\rho_{\rm e}^{1/2}v\right] = 0,$$

$$v(k,t) = \int_{-\infty}^{\infty} {\rm e}^{-{\rm i}kx} V(x,t){\rm d}x. \qquad (23)$$

Equation (23) is identical with the equation for a parametrically perturbed pendulum. We note that the simplest of such equations, Mathieu's equation (Blanch 1972) does indeed exhibit instability, provided the parametric perturbation occurs at a suitable frequency. In our wave problem, the frequencies of the perturbation are rather randomly distributed: our investigation shows that, when this distribution is Gaussian, waves are always driven unstable. This has an intuitive explanation if we think of a child wiggling on a swing.

We now come to the more technical part of our work. In analyzing the real part of the frequency shift, wave acceleration was already reported in the literature. For instance, Stix & Zhugzhda (1998) investigated corrections to the frequencies of p-modes due to the sound speed and velocity inhomogeneity of the convection zone. In most cases the frequency shifts were negative, although for the spherical degrees $l = 1200$ and $l = 1400$ these shifts were positive for the modes p_1, p_2, and p_3. On the other hand, Rüdiger et al. (1997) showed that the Reynolds stress shifts up the frequencies while the density fluctuations lead to frequency decrease. These two competitive effects shift up low frequencies and reduce high frequencies.

The discrepancy between our results and the results by other authors are mainly due to the difference in the physical processes being analyzed. Among the major sources of diversity we can mention different wave models (e.g. Stix & Zhugzhda 1998), different boundary conditions (e.g. Stein & Nordlund 1998), and different properties of the random perturbations (e.g. Rüdiger et al. 1997). It is noteworthy here that a random density field is not the only effect which can account for the departure of the observational data from the theoretical data. An alternative explanation is based on a wave tunneling through the chromospheric barrier Dzhalilov et al. (2000). Such tunneling leads to an increase of low frequencies and decrease of high frequencies.

Concerning the imaginary part of the frequency shift, we stress that, while in the case of time independent perturbations (Nocera et al. 2001) an overstability occurred, due a virtual root of the dispersion function lying in the lower half of the dispersion's *unphysical* Riemann sheet, in this work we have a genuine instability: indeed the root found in Eqs. (13) and (17) lies in the upper half part of the *physical* sheet. This violation of wave energy conservation is due to the source term in Eq. (1).

Puzzling as it may be, the instability thus found is against the observed stability of p-modes; we might draw the conclusion that the envisaged Gaussian correlation function of Eq. (11) is unrealistic. A more instructive conclusion, however is that sources for the random mass fluctuations which depend only on time should be considered carefully, as much as those which are time-independent. In the more realistic space- and time-dependent randomness scenario, the overstability found by Nocera et al. (2001) and the instability found in the present work will possibly compete. The resulting balance of these effects will be dealt with elsewhere.

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