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On the apsidal motion in close binaries due to the tidal deformations of the components

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Abstract. The paper is devoted to a confrontation of the apsidal-motion rates in close binaries due to the tidal perturbations of the stellar components that are predicted by Sterne's formula (1939) with the corresponding apsidal-motion rates that are determined in the framework of the theory of the dynamic tides. Sterne's formula is derived in the supposition that the orbital period and the star's rotational period are sufficiently long so that, in accordance with an earlier suggestion of Cowling (1938), the star is almost adjusted to the gravitational field of the companion. From the point of view of the theory of the dynamic tides, the second-degree tide is then approximated at each instant by an appropriate linear combination of three second-degree tides which are considered to be static. In this limiting case, the rate of secular apsidal motion predicted by Sterne's formula agrees, up to large orbital eccentricities, with the rate of secular apsidal motion determined in the framework of the theory of the dynamic tides and depends on the star's central mass condensation. For close binaries with shorter orbital periods, the use of Sterne's formula leads to deviations because of the increasing influence of the compressibility of the stellar fluid and resonances of dynamic tides with lower-order g^+-modes. The relative deviations may to amount to a few tens of percents for models of zero-age main sequence stars of $5\,M_\odot$, $10\,M_\odot$, and $20\,M_\odot$.

Key words. stars: binaries: close – stars: oscillations – celestial mechanics – stellar dynamics

1. Introduction

A formula for the apsidal motion in close binaries due to the tidal deformations of the components was first derived in 1928 by Russell. Later, Cowling (1938) observed that Russell derived his formula under the ideas that the components of a close binary can be regarded as ellipsoids whose longest axes are directed along the line joining their centres of mass, and that the forms of the two stars are invariant while their tidal distortions actually vary with the varying distance between them. On the basis of an investigation with regard to two incompressible masses with uniform densities which are moving around each other in nearly circular orbits, Cowling suggested that the form of a star that is subject to the gravitational field of a companion is at any time close "to the *equilibrium* form in which it is completely adjusted to the gravitational field of its companion". He also derived a formula for the apsidal motion.

Sterne (1939), on his side, studied the apsidal motions in close binaries "in the limiting case where the orbital period is long compared with the free harmonic periods of the component stars". The component stars were considered to be compressible fluids. In a footnote, he added:

> The present treatment is essentially along the lines laid down by Russell... except that here (a) the variation of the tidal deformation with distance between centres will be taken into account, and (b) a higher order of accuracy will be aimed at. The present treatment differs greatly from the treatment given by Cowling..., with whose results, however, to Cowling's order of accuracy, it agrees.

He also observed:

> The results should be applicable in the absence of approaches to resonance so close as to render necessary the more general dynamical theory.

Sterne's formula has become the standard formula for the theoretical determination of the apsidal motion in close binaries due to the tidal deformations of the two components.

This investigation is devoted to a confrontation of the rates of secular apsidal motion predicted by Sterne's

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formula with those determined in the framework of the theory of dynamic tides.

The plan of the paper is as follows. In Sect. 2, the basic assumptions are presented. In Sect. 3, Sterne's theory for the apsidal motion due to the tidal deformation of the components is reconsidered in the light of the theory of the dynamic tides. In Sect. 4, we show the validity of Sterne's formula for close binaries in which both the orbital period of the components and the period of stellar rotation are long. In Sect. 5, we examine the deviations that appear between the rates of secular apsidal motion predicted by Sterne's formula and those determined within the framework of the theory of the dynamic tides for close binaries with shorter orbital periods. In the final section, concluding remarks are presented.

2. Basic assumptions

Consider a close binary system of stars which are orbiting around each other under the influence of their mutual gravitational force. We assume the first star, with mass M_1 and radius R_1, to rotate uniformly around an axis perpendicular to the orbital plane, with an angular velocity $\boldsymbol{\Omega}$ in the sense of the orbital motion. We neglect the effects of the Coriolis force and the centrifugal force, so that the tides raised by the companion correspond to those of a non-rotating spherically symmetric star. The companion, with mass M_2, is treated as a point mass.

With the centre of mass of the first star, we let coincide the origin of an orthogonal frame of reference that is corotating with the star. The z-axis is normal to the orbital plane and is oriented in the sense of the star's angular velocity. With respect to this frame of reference, we introduce the spherical coordinates $\boldsymbol{r} = (r, \theta, \phi)$.

Let u be the distance between the two components, v the true anomaly of the companion in its relative orbit, a the semi-major axis of the relative orbit, and e the orbital eccentricity. Furthermore, let $\varepsilon_{\rm T}$ be a small parameter defined as the ratio of the tidal force to the gravity at the star's equator:

$$\varepsilon_{\rm T} = \left(\frac{R_1}{a} \right)^3 \frac{M_2}{M_1}. \tag{1}$$

We consider the tide-generating potential to be expanded in terms of unnormalised spherical harmonics $Y_\ell^m(\theta, \phi)$ and of multiples of the companion's mean motion n as

$$\varepsilon_{\rm T}\, W(\boldsymbol{r}, t) = -\varepsilon_{\rm T} \frac{G\, M_1}{R_1} \sum_{\ell=2}^{4} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} c_{\ell,m,k} \left(\frac{r}{R_1} \right)^{\ell} Y_\ell^m(\theta, \phi)\, \exp\left[{\rm i}\left(\sigma_{\rm T}\, t - k\, n\, \tau \right) \right], \tag{2}$$

where G is the Newtonian constant of gravitation, $\sigma_{\rm T}$ the forcing angular frequency with respect to the corotating frame of reference given by

$$\sigma_{\rm T} = k\, n + m\, \Omega, \tag{3}$$

and τ a time of periastron passage. The coefficients $c_{\ell,m,k}$ are Fourier coefficients determined as

$$c_{\ell,m,k} = \frac{(\ell - |m|)!}{(\ell + |m|)!} P_\ell^{|m|}(0) \left(\frac{R_1}{a} \right)^{\ell-2} \frac{1}{\left(1-e^2\right)^{\ell-1/2}} \frac{1}{\pi} \int_0^{\pi} (1 + e\, \cos v)^{\ell-1} \cos(k\, M + m\, v)\, {\rm d}v. \tag{4}$$

In this expression, $P_\ell^{|m|}(x)$ is an associated Legendre polynomial of the first kind, and M the mean anomaly of the star's companion.

Each term in the expansion of the tide-generating potential given by Eq. (2) gives rise to a partial tide in the rotating star: the terms associated with $m = 0$ and $k = 0$ lead to *static* tides, the other terms to *dynamic* tides. With each degree ℓ of the spherical harmonics $Y_\ell^m(\theta, \phi)$, one static tide is associated. An *equilibrium* tide is generated in a star when the orbit of the companion is circular, and the star rotates synchronously with the orbital motion of the companion.

The Fourier coefficients $c_{\ell,m,k}$ depend on the orbital eccentricity. They decrease in absolute value as k increases. When the orbital eccentricity is large, the decrease is slow, so that many partial tides must be taken into consideration in the determination of the tidal distortion caused by the star's companion.

Hereafter, we restrict ourselves to second-degree tides.

3. Sterne's theory of the rate of apsidal motion

The second-degree tide-generating potential at a point with spherical coordinates r, θ, ϕ can be written as

$$\varepsilon_{\rm T}\, W_2(\boldsymbol{r}, t) = -\varepsilon_{\rm T} \frac{G\, M_1}{R_1} \left(\frac{r}{R_1} \right)^2 \left\{ -\frac{1}{2} \left(\frac{u}{a} \right)^{-3} Y_2(\theta) + \frac{1}{8} \left(\frac{u}{a} \right)^{-3} \exp[2\,{\rm i}\,(\Omega\, t - v)]\, Y_2^2(\theta, \phi) + \frac{1}{8} \left(\frac{u}{a} \right)^{-3} \exp[-2\,{\rm i}\,(\Omega\, t - v)]\, Y_2^{-2}(\theta, \phi) \right\}. \tag{5}$$

In the limiting cases of *long orbital periods*, the distance u between the two components and the true anomaly v of the star's companion vary slowly with time. When moreover the star's *angular velocity of rotation is small*, the coefficients

$$\left. \begin{aligned} c_{2,0}^* &= -\frac{1}{2} \left(\frac{u}{a} \right)^{-3}, \\ c_{2,2}^* &= \frac{1}{8} \left(\frac{u}{a} \right)^{-3} \exp[2\,{\rm i}\,(\Omega\, t - v)], \\ c_{2,-2}^* &= \frac{1}{8} \left(\frac{u}{a} \right)^{-3} \exp[-2\,{\rm i}\,(\Omega\, t - v)] \end{aligned} \right\} \tag{6}$$

also vary slowly with time. The coefficients $c_{2,0}^*$, $c_{2,2}^*$, $c_{2,-2}^*$ generally differ from the second-degree Fourier coefficients $c_{2,0,0}$, $c_{2,2,-2}$, $c_{2,-2,2}$, which, in the case of an equilibrium tide, are the only second-degree Fourier coefficients different from zero and have the values

$$c_{2,0,0} = -\frac{1}{2}, \quad c_{2,2,-2} = \frac{1}{8}, \quad c_{2,-2,2} = \frac{1}{8}. \tag{7}$$

On the basis of Cowling's suggestion that the form of the star is at any time close "to the *equilibrium* form in which it is completely adjusted to the gravitational field of its companion", the time-dependent tide in the star is approximated by a linear combination of three second-degree *static* tides associated with the instantaneous values of the coefficients $c^*_{2,0}$, $c^*_{2,2}$, $c^*_{2,-2}$. The linear combination of the three static tides becomes a true equilibrium tide in the limiting case in which the eccentricity of the companion's orbit is equal to zero and the star's rotation is synchronized with the companion's orbital motion, since one then has that

$$u = a, \quad v = M, \quad \Omega\, t = M. \tag{8}$$

For the static tide of degree ℓ, the radial part of the radial component of the tidal displacement, $\xi_{\mathrm{st},\ell}(r)$, is the solution of the homogeneous second-order differential equation

$$\frac{\mathrm{d}^2 \xi_{\mathrm{st},\ell}}{\mathrm{d}r^2} + 2\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{1}{r}\right)\frac{\mathrm{d}\xi_{\mathrm{st},\ell}}{\mathrm{d}r} - \frac{\ell(\ell+1)-2}{r^2}\,\xi_{\mathrm{st},\ell} = 0 \tag{9}$$

that remains finite at $r = 0$ and satisfies the boundary condition

$$\left(\frac{\mathrm{d}\xi_{\mathrm{st},\ell}}{\mathrm{d}r}\right)_{R_1} + \frac{\ell-1}{R_1}\left(\xi_{\mathrm{st},\ell}\right)_{R_1} = \varepsilon_{\mathrm{T}}\,(2\,\ell+1)\,c_{\ell,0,0} \tag{10}$$

at $r = R_1$. In Eq. (9), g is the local gravity.

In Sterne's theory, second-degree static solutions are considered for which the second-degree Fourier coefficient $c_{2,0,0}$ in the boundary condition given by Eq. (10) is replaced successively by the instantaneous value of the slowly varying coefficients $c^*_{2,0}$, $c^*_{2,2}$, $c^*_{2,-2}$. Hence, the solutions can be cast in the form

$$\xi_{\mathrm{st};2,m}(r) = \varepsilon_{\mathrm{T}}\, c^*_{2,m}\, \xi^*_{\mathrm{st},2}(r), \quad m = -2, 0, 2. \tag{11}$$

For static tides, the radial component of the tidal displacement is related to the sum of the Eulerian perturbation of the gravitational potential and the tide-generating potential. From this relation, it follows that the Eulerian perturbation of the gravitational potential associated with a solution for the radial component of the tidal displacement is given by

$$\Phi'_{\mathrm{st};2,m}(r,\theta,\phi) = -\left[g\,\xi_{\mathrm{st};2,m}(r) - \varepsilon_{\mathrm{T}}\,\frac{G\,M_1}{R_1}\,c^*_{2,m}\left(\frac{r}{R_1}\right)^2\right] Y_2^m(\theta,\phi), \quad m = -2, 0, 2. \tag{12}$$

By the introduction of the constant k_2 as

$$2\,k_2 = \frac{\xi^*_{\mathrm{st},2}\,(R_1)}{R_1} - 1, \tag{13}$$

the Eulerian perturbation of the gravitational potential at the star's surface can be expressed as

$$\Phi'_{\mathrm{st};2,m}\,(R_1,\theta,\phi) = -\varepsilon_{\mathrm{T}}\,\frac{G\,M_1}{R_1}\,2\,k_2\,c^*_{2,m}\,Y_2^m(\theta,\phi), \quad m = -2, 0, 2. \tag{14}$$

The Eulerian perturbation of the external gravitational potential is a solution of Laplace's equation. The solution associated with the spherical harmonic $Y_2^m(\theta,\phi)$ that tends to zero as $r \to \infty$ can be written as

$$\Phi'_{\mathrm{e}}\,(\boldsymbol{r}\,) = K\left(\frac{r}{R_1}\right)^{-3} Y_2^m(\theta,\phi). \tag{15}$$

Here K is a constant which is determined by the requirement that the Eulerian perturbation of the gravitational potential be continuous at the star's surface. The solution given in Eq. (15) then takes the form

$$\Phi'_{\mathrm{e}}\,(\boldsymbol{r}\,) = -\varepsilon_{\mathrm{T}}\,\frac{G\,M_1}{R_1}\,2\,k_2\,c^*_{2,m}\left(\frac{r}{R_1}\right)^{-3} Y_2^m(\theta,\phi), \quad m = -2, 0, 2. \tag{16}$$

The spherical coordinates of the companion with respect to the corotating frame of reference are, at any instant t,

$$r = u, \quad \theta = \pi/2, \quad \phi = v - \Omega\, t. \tag{17}$$

Consequently, with the use of the definition of the small parameter ε_{T} given in Eq. (1), the global Eulerian perturbation of the external gravitational potential at the position of the star's companion can be written as

$$\Phi'_{\mathrm{e}}\left(u, \frac{\pi}{2}, v - \Omega\, t; t\right) = -\frac{G\,M_1}{R_1}\left(\frac{R_1}{a}\right)^6 \frac{M_2}{M_1}\,2\,k_2\left(\frac{u}{a}\right)^{-3} \Big\{c^*_{2,0}\,P_2(0) + \big\{c^*_{2,2}\,\exp[2\,\mathrm{i}\,(v-\Omega\,t)] + c^*_{2,-2}\,\exp[-2\,\mathrm{i}\,(v-\Omega\,t)]\big\}\,P_2^2(0)\Big\}. \tag{18}$$

On the grounds of perturbation theory of celestial mechanics, the rate of change of the longitude of the periastron is given by

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = \frac{1}{n\,a^2}\,\frac{\left(1-e^2\right)^{1/2}}{e}\,\frac{\partial R}{\partial e}, \tag{19}$$

where R is the perturbing function (see, e.g., Smart 1953; Sterne 1960; Brouwer & Clemence 1961). This function is related to the Eulerian perturbation of the gravitational potential at the position of the companion as

$$R = -\frac{M_1 + M_2}{M_1}\,\Phi'_{\mathrm{e}}\left(u, \frac{\pi}{2}, v - \Omega\,t; t\right) \tag{20}$$

(Smeyers et al. 1991). Equation (19) leads to

$$\frac{\mathrm{d}\varpi}{\mathrm{d}M} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\,\frac{\left(1-e^2\right)^{1/2}}{e}\,6\,k_2\left(\frac{u}{a}\right)^{-7}\cos v. \tag{21}$$

Averaging over a revolution of the companion then yields the equation for the rate of secular change of the longitude of the periastron given by Sterne

$$\left(\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right)_{\mathrm{standard}} \equiv \frac{n}{2\,\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\varpi}{\mathrm{d}M}\,\mathrm{d}M = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\,\frac{2\,\pi}{T_{\mathrm{orb}}}\,k_2\,15\,f\left(e^2\right), \tag{22}$$

where $T_{\rm orb}$ is the orbital period. The function $f\left(e^2\right)$ is defined as

$$f\left(e^2\right) = \left(1-e^2\right)^{-5} \left(1 + \frac{3}{2}\,e^2 + \frac{1}{8}\,e^4\right). \tag{23}$$

It is always positive and increases monotonically with the orbital eccentricity.

According to Sterne's theory, the apsidal motion in a close binary due to a star's tidal deformation is always an apsidal advance. The constant k_2 is called the apsidal-motion constant and depends on the mass concentration of the tidally distorted star: it takes the value 0.75 in the limiting case of an equilibrium sphere with uniform mass density, and the value 0 in the limiting case of a Roche model in which the star is reduced to a point mass.

4. The validity of Sterne's theory for long orbital and rotational periods

Within the framework of the theory of the dynamic tides, the apsidal motion due to a star's tidal deformation results from the various partial tides that are generated by the companion in the star. According to Smeyers et al. (1998), the total rate of secular apsidal motion due to the second-degree tides can be expressed in terms of functions $G_{2,m,k}(e)$ of the orbital eccentricity, and of coefficients $F_{2,m,k}$ which render the star's responses to the various forcing frequencies $\sigma_{\rm T}$. In the derivation of the expression, the partial tides are treated as forced isentropic oscillations of the spherically symmetric star.

The equation for the rate of secular apsidal motion due to a star's tidal deformation takes the form

$$\left(\frac{{\rm d}\varpi}{{\rm d}t}\right)_{\rm dyn} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\,\frac{2\,\pi}{T_{\rm orb}} \left[2\,k_2\,G_{2,0,0} + 4\sum_{k=1}^{\infty} \left(F_{2,0,k}\,G_{2,0,k} + F_{2,2,k}\,G_{2,2,k} + F_{2,-2,k}\,G_{2,-2,k}\right)\right], \tag{24}$$

where the functions $G_{2,m,k}(e)$ are determined as

$$G_{2,m,k}(e) = \frac{1}{e\,\left(1-e^2\right)^2}\,c_{2,m,k}\,P_2^{|m|}(0) \, \frac{1}{\pi}\left[3\int_0^{\pi} (1+e\,\cos v)^2\,\cos(m\,v + k\,M)\,\cos v\,{\rm d}v - m\int_0^{\pi} (1+e\cos v)\,(2+e\cos v) \sin(m\,v+k\,M)\,\sin v\,{\rm d}v\right]. \tag{25}$$

In particular, the function $G_{2,0,0}(e)$ is given by

$$G_{2,0,0}(e) = \frac{3}{4}\left(1-e^2\right)^{-7/2}. \tag{26}$$

The coefficients $F_{2,m,k}$ are determined as

$$2\,F_{2,m,k} = -\left[\frac{R_1}{G\,M_1}\,\frac{\Psi_{\rm T}\left(R_1\right)}{\varepsilon_{\rm T}\,c_{2,m,k}} + 1\right], \tag{27}$$

where $\Psi_{\rm T}\left(R_1\right)$ is the sum of the radial parts of the tide-generating potential and the Eulerian perturbation of the star's gravitational potential at $r = R_1$. They change with the companion's orbital period and the star's rotational period. In the limiting case in which $T_{\rm orb} \rightarrow \infty$ and $\Omega \rightarrow 0$, all forcing angular frequencies $\sigma_{\rm T}$ tend to zero, and the constants $F_{2,m,k}$ tend to the apsidal-motion constant k_2:

$$\lim_{T_{\rm orb}\rightarrow\infty,\,\Omega\rightarrow 0} F_{2,m,k} = k_2, \quad m = -2, 0, 2, \;\; k = 0, 1, 2, \ldots \tag{28}$$

The rate of secular apsidal motion is then given by

$$\lim_{T_{\rm orb}\rightarrow\infty,\,\Omega\rightarrow 0} \left(\frac{{\rm d}\varpi}{{\rm d}t}\right)_{\rm dyn} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\,\frac{2\,\pi}{T_{\rm orb}}\,k_2\,2\,g(e), \tag{29}$$

where the function $g(e)$ is determined as

$$g(e) = G_{2,0,0}(e) + 2\sum_{k=1}^{\infty} \left[G_{2,0,k}(e) + G_{2,2,k}(e) + G_{2,-2,k}(e)\right]. \tag{30}$$

We have compared the variation of Sterne's function $15\,f\left(e^2\right)$, which appears in Eq. (22), with the variation of the function $2\,g(e)$ in two ways. First, we derived a Taylor series for the function $2\,g(e)$ that is exact to the sixth order in the orbital eccentricity. For this purpose, we used Taylor series for the functions $G_{2,m,k}(e)$ associated with $m = -2, 0, 2$ and $k = 0, 1, 2, \ldots$ (Willems 2000). The resulting Taylor series takes the form

$$2\,g(e) = 15\left(1 + \frac{13}{2}\,e^2 + \frac{181}{8}\,e^4 + \frac{465}{8}\,e^6\right). \tag{31}$$

The Taylor series for Sterne's function $15\,f\left(e^2\right)$ considered to the same order of approximation in the orbital eccentricity agrees with the Taylor series for the function $2\,g(e)$.

Secondly, we compared the variation of the function $15\,f\left(e^2\right)$ with that of the function $2\,g(e)$ numerically up to the value of the orbital eccentricity $e = 0.9$. Here too, we found an excellent agreement of the function $15\,f\left(e^2\right)$ with the function $2\,g(e)$.

Hence, as $T_{\rm orb} \rightarrow \infty$ and $\Omega \rightarrow 0$, the rate of secular apsidal motion predicted by Sterne's formula agrees, up to *large orbital eccentricities*, with the rate of secular apsidal motion established within the framework of the theory of the dynamic tides.

5. Deviations of Sterne's theory for shorter orbital periods

For close binaries with shorter orbital periods, deviations of the rate of secular apsidal motion predicted by Sterne's formula in comparison with the rate of secular apsidal motion given by the theory of the dynamic tides can be expected. Quataert et al. (1996) already observed that the influence of dynamic tides on the rate of secular apsidal motion is important when the orbital period is not sufficiently long in comparison with the periods of the star's free oscillation modes.

Deviations may be expected for two reasons. First, the coefficients $F_{2,m,k}$ can no longer be approximated by the apsidal-motion constant k_2 and must be determined by Eq. (27). A main physical reason is that the *compressibility* of the stellar fluid will play a more important role in the star's response to the forcing angular frequency as this frequency becomes larger in absolute value. The coefficients $F_{2,m,k}$ can then be negative as well as positive.

Secondly, *resonances* of dynamic tides with free oscillation modes g^+ may occur. Resonances of a dynamic tide with a free oscillation mode in a component of a close binary have been described by Smeyers et al. (1998) by means of a two time-variable expansion procedure, in which the resonant dynamic tides are treated in the isentropic approximation. In a more recent investigation, Willems & Smeyers (2001) have extended the two time-variable expansion procedure in order to determine the nonadiabatic effects due to the radiative energy flux on resonant dynamic tides in a component of a close binary. These nonadiabatic effects are left out of consideration here, since they lead only to a second-order effect as far as the rate of secular apsidal motion is concerned.

Consider a dynamic tide that has a forcing angular frequency $\sigma_{\rm T}$ close to the eigenfrequency $\sigma_{2,N}$ of a free second-degree oscillation mode of radial order N. Let the relative frequency difference

$$\varepsilon = \frac{\sigma_{2,N} - \sigma_{\rm T}}{\sigma_{2,N}} \tag{32}$$

be of the order of the ratio $\varepsilon_{\rm T}$ of the tidal force to the gravity at the star's equator. Then, at the lowest order of approximation in the perturbation theory, the coefficient $F_{2,m,k}$ in the right-hand member of Eq. (24) that is associated with the resonant dynamic tide is determined as

$$F_{2,m,k} = \frac{1}{4\,\varepsilon}\,H_{2,N}. \tag{33}$$

The factor $H_{2,N}$ is a constant factor depending solely on the second-degree oscillation mode of radial order N that is involved in the resonance. The variation of the factor $H_{2,N}$ as a function of the radial order N of the mode is displayed in Fig. 3 of Smeyers et al. (1998) for the polytropic models with indices $n_{\rm e} = 2, 3, 4$ and for the value 5/3 of the generalized isentropic coefficient $\Gamma_1 \equiv (\partial \ln P/\partial \ln \rho)_S$. The factor $H_{2,N}$ decreases very rapidly as the radial order of the g^+-mode increases. The decrease is slower for a polytropic model with a larger index. From the rapid decrease of the factor $H_{2,N}$, it results that g^+-modes of higher radial orders have no noticeable impact on the rate of secular apsidal motion, so that only g^+-modes of *low* radial orders must be taken into consideration.

We have examined the extent of the deviations generated by the use of Sterne's formula for the rate of secular apsidal motion in close binaries with shorter orbital periods. To this end, we used zero-age main sequence stellar models of $5\,M_\odot$, $10\,M_\odot$, and $20\,M_\odot$ consisting of a convective core and a radiative envelope. The ages of the stars, the radii R_1, the ratios of the central mass density $\rho_{\rm c}$ to the mean mass density $\overline{\rho}$, and the apsidal-motion constants k_2 are listed in Table 1. We considered orbital periods ranging from 2 to 6 days and the orbital eccentricities $e = 0.25$ and $e = 0.5$. For the determination of the forcing angular frequencies $\sigma_{\rm T}$ with respect to the corotating frame of reference, we adopted the low angular frequency of rotation $\Omega = 0.01\,n$, so that the period of the companion's motion relative to the rotating star remains short.

Table 1. Properties of the zero-age main sequence stellar models of $5\,M_\odot$, $10\,M_\odot$, and $20\,M_\odot$.

Mass	age (yrs)	R_1 (km)	$\rho_{\rm c}/\overline{\rho}$	k_2
$5\,M_\odot$	3.03×10^5	1.93×10^6	61.8	0.00742
$10\,M_\odot$	1.38×10^5	2.76×10^6	39.4	0.0119
$20\,M_\odot$	3.97×10^4	4.05×10^6	32.5	0.0155

With the parameters chosen, we determined the relative difference between the rate of secular apsidal motion predicted by Sterne's formula and the corresponding rate given by the theory of the dynamic tides:

$$\Delta = \frac{({\rm d}\varpi/{\rm d}t)_{\rm standard} - ({\rm d}\varpi/{\rm d}t)_{\rm dyn}}{({\rm d}\varpi/{\rm d}t)_{\rm dyn}}. \tag{34}$$

In the perturbation theory used by Smeyers et al. (1998), the relative frequency difference ε is assumed to be of the order of the expansion parameter $\varepsilon_{\rm T}$. Therefore, in the determination of the relative differences Δ, we limited the values of ε to values larger than or equal to $0.1\,\varepsilon_{\rm T}$ near resonances. The results are represented by the solid lines in Figs. 1–3.

For the three zero-age main sequence stellar models, the solid lines representing the variation of the relative difference Δ appear to be the result of a superposition of numerous peaks, positive and negative, on a continuous curve lying below the horizontal axis. The basic curve renders the systematic deviations of the rate of secular apsidal motion predicted by Sterne's formula that would result from the changes in the star's responses to the companion's tidal action in the absence of resonances of dynamic tides with free oscillation modes g^+. The underlying physical reason for the changes in the star's response at shorter orbital periods is the growing role of the compressibility of the stellar fluid. The peaks superposed on the basic curve render the additional deviations of the rate of secular apsidal motion predicted by Sterne's formula that stem from the resonances of dynamic tides with free oscillation modes g^+ of lower radial orders.

The basic curves can be approximately represented by an equation of the form

$$\Delta_{\rm basic} = -\lambda\,T_{\rm orb}^{-2}, \tag{35}$$

where λ is a constant. For the determination of the constant λ, we adopted a linear least-squares procedure described by Press et al. (1992) and fitted Eq. (35) to the

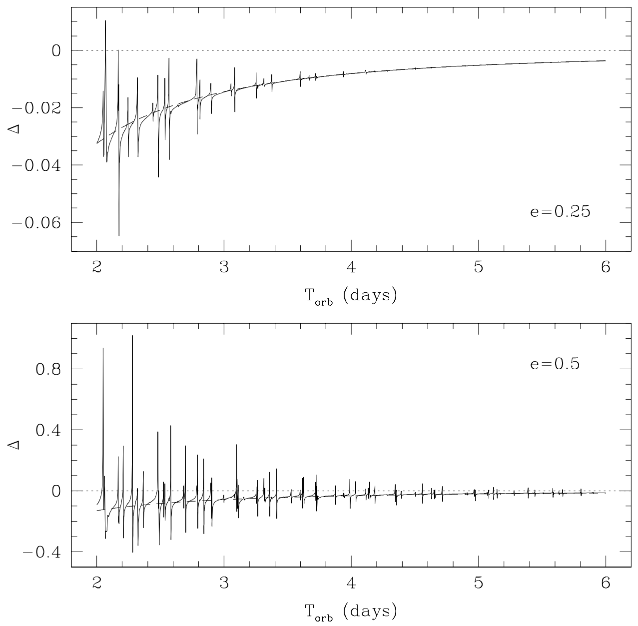

Fig. 1. Relative differences Δ for a 5 $M_\odot$ ZAMS star, in presence (solid line) and in absence (dashed line) of resonances of dynamic tides with free oscillation modes g^+ of lower radial orders.

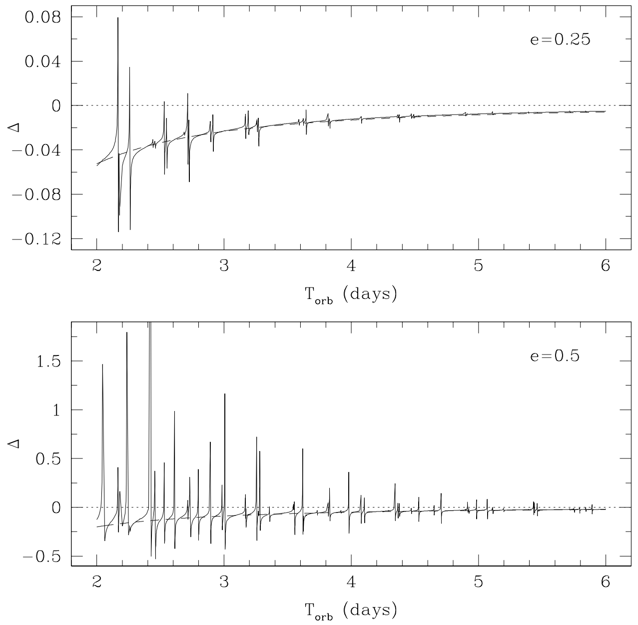

Fig. 2. Relative differences Δ for a 10 $M_\odot$ ZAMS star, in presence (solid line) and in absence (dashed line) of resonances of dynamic tides with free oscillation modes g^+ of lower radial orders.

relative differences Δ displayed in Figs. 1–3. With the orbital period expressed in days, the procedure leads to the values of λ listed in Table 2. The resulting values have been used for the construction of the dashed lines in Figs. 1–3.

The downwards slope of the basic curve is larger for a model with a larger mass. This property is related to the stronger impact of the tidal action on a less centrally condensed stellar model. For a given stellar model, the

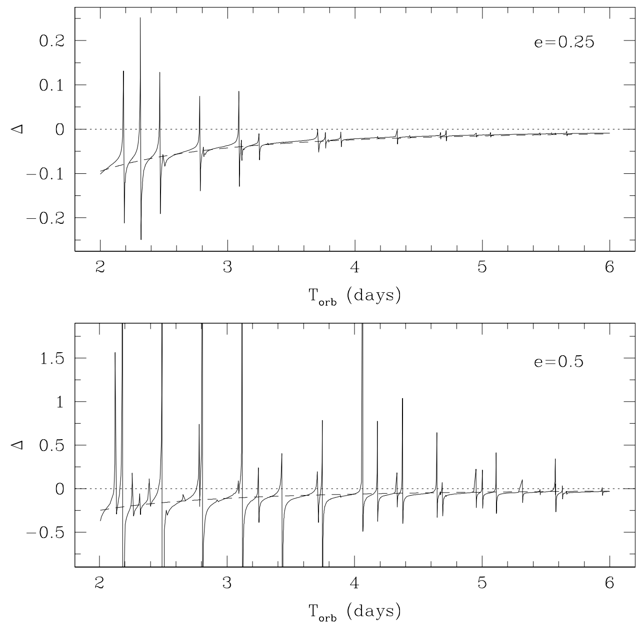

Fig. 3. Relative differences Δ for a 20 $M_\odot$ ZAMS star, in presence (solid line) and in absence (dashed line) of resonances of dynamic tides with free oscillation modes g^+ of lower radial orders.

Table 2. The proportionality factor λ for the 5 $M_\odot$, the 10 $M_\odot$, and the 20 $M_\odot$ ZAMS stellar model and the orbital eccentricities $e = 0.25$ and $e = 0.5$.

Mass	$\lambda_{e=0.25}$	$\lambda_{e=0.5}$
5 $M_\odot$	0.13	0.52
10 $M_\odot$	0.21	0.80
20 $M_\odot$	0.38	1.0

distance of the basic curve to the horizontal axis increases with the orbital eccentricity.

In the absence of resonances, Sterne's formula for the rate of secular apsidal motion leads systematically to *negative* deviations from the corresponding rate determined by the theory of the dynamic tides. Therefore, it yields somewhat too small values for the rate of secular apsidal motion, and thus *somewhat too long apsidal-motion periods*. With the small angular velocity of the star's rotation chosen, the systematic relative difference amounts to as much as 25% at the shortest orbital periods for the zero-age main sequence stellar model of 20 $M_\odot$ and the orbital eccentricity $e = 0.5$.

Upon the systematically negative deviations, large periastron *advances* and strongly counteracting periastron *recessions* are superposed by the resonances of dynamic tides with lower-order g^+-modes. Consequently, because of the resonances, the differences of the rates of secular apsidal motion derived by means of Sterne's formula may be partly or almost completely annihilated or may be augmented to as much as several tens of percents.

For a given orbital eccentricity, more resonances of dynamic tides with lower-order g^+-modes occur in the 5 $M_\odot$ model than in the 10 $M_\odot$ and the 20 $M_\odot$ model. The larger

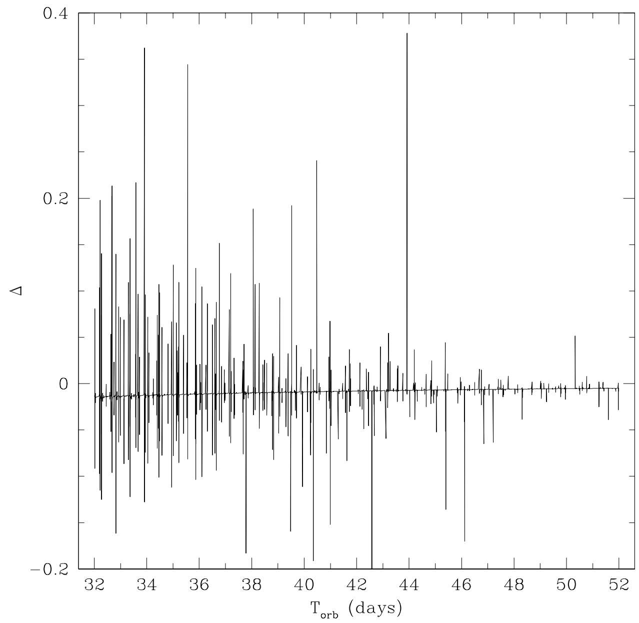

Fig. 4. The variations of the relative difference Δ as a function of the orbital period, for the $10\,M_{\odot}$ ZAMS model and the orbital eccentricity $e = 0.808$ corresponding to the orbital eccentricity of the radio pulsar PSR J0045-7319.

number of resonances is related to the reduction of the size of the radiative envelope which causes the eigenfrequencies of the $5\,M_{\odot}$ model to be smaller and closer to each other than those of the $10\,M_{\odot}$ and the $20\,M_{\odot}$ model.

The rate of secular apsidal motion generated by a tidally distorted $10\,M_{\odot}$ star on an orbit with the high eccentricity $e = 0.808$ has been considered by Kumar et al. (1995) relative to the radio pulsar PSR J0045-7319. The authors determined the rate of secular apsidal motion due to the dynamic tides and concluded that, for the longer orbital period of 51.169 days involved, their result was in excellent agreement with the classical apsidal-motion formula. In view of the very high orbital eccentricity, the rate of secular apsidal motion must be strongly influenced by numerous resonances of dynamic tides. Therefore, we found it worthwhile to redetermine the rate of secular apsidal motion generated by the $10\,M_{\odot}$ ZAMS stellar model for the eccentricity $e = 0.808$. The resulting variation of the relative difference Δ as a function of the orbital period is presented in Fig. 4 for the range of orbital periods from 32 to 52 days. Even for such long orbital periods, numerous resonances occur which lead to relative differences up to ten percent and more. For orbital periods near 51.169 days, the relative difference Δ is reduced to -0.5%, but still some small peaks due to resonances of dynamic tides with g^+-modes appear.

6. Concluding remarks

In this paper, we have examined the validity of Sterne's formula for the secular apsidal motion in close binaries due to the tidal deformations of the stellar components.

From the point of view of the theory of the dynamic tides, Sterne's formula is derived in the supposition that the orbital period and the star's rotational period are sufficiently long for the tide generated by a companion that is moving in an eccentric orbit to be approximated at each instant by an appropriate linear combination of three static tides. Under these conditions, Sterne's formula for the rate of secular apsidal motion is valid up to high orbital eccentricities. For a given orbital eccentricity, the secular apsidal motion generated by a tidally perturbed star then depends exclusively on the central condensation of the star's mass.

For shorter orbital periods, the rate of secular apsidal motion predicted by Sterne's formula deviates from the rate of secular apsidal motion determined within the framework of the theory of the dynamic tides for two reasons.

First, as forced oscillations, dynamic tides in a star depend not only on the gravity and the density stratification in the various stellar layers but also on the compressibility of these layers. The dependency on the compressibility of the stellar fluid becomes larger for higher-frequency dynamic tides. As the orbital period becomes shorter, the change in a star's response to the tidal action of a companion leads to an increasing deviation of the rate of secular apsidal motion predicted by Sterne's formula from the rate of secular apsidal motion given by the theory of the dynamic tides. For the zero-age main sequence stellar models of $5\,M_{\odot}$, $10\,M_{\odot}$, and $20\,M_{\odot}$ considered, the deviation is systematically negative and approximately inversely proportional to the square of the orbital period. For the small angular velocity of stellar rotation adopted, the deviations amount to as much as 25% at the shortest orbital periods in the case of the orbital eccentricity $e = 0.5$. Consequently, Sterne's formula yields apsidal-motion periods somewhat longer than those that are derived in the framework of the theory of the dynamic tides.

Secondly, Sterne's formula for the rate of secular apsidal motion leads to deviations because of resonances of dynamic tides with lower-order g^+-modes which occur at various shorter orbital periods. The number of orbital periods at which resonances occur, grows with the orbital eccentricity. Due to the resonances, the negative deviations of the rates of secular apsidal motion derived by means of Sterne's formula may be partly or almost completely annihilated or may be augmented to as much as several tens of percents.

In close binaries with rapidly rotating stars, the period of the companion's motion relative to the rotating star will be longer. Therefore, the deviations of Sterne's formula for the rate of secular apsidal motion may be expected to be reduced in comparison to the deviations found in the assumption of a slow angular velocity of stellar rotation.

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