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Hamiltonian theory for the non–rigid Earth: Semidiurnal terms

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Abstract. The purpose of this paper is to determine the contributions to the nutation series arising from the triaxiality of a non-rigid Earth model composed of a rigid mantle and a liquid core. With this aim, the canonical formulation of the rotation of the non-rigid Earth developed by Getino and Ferrándiz is applied in order to study the semidiurnal terms arising from the C_{22} and S_{22} geopotential coefficients. Once the corresponding generating function is calculated, analytical expressions of the Andoyer and figure planes are derived. We also provide numerical nutation series based on the analytical formulae.

Key words. celestrial mechanics – earth – method: analytical

1. Introduction

The Hamiltonian formalism has been applied by Getino and Ferrándiz to the study of non–rigid Earth models. At present, we have performed a comprehensive treatment of an Earth model composed of an elastic, axis–symmetric mantle, and a liquid core, including the dissipative effects in the core–mantle boundary (CMB) and the delay in the response of the deformation due to the inelasticity of the mantle. The corresponding free problem, with the determination of the free frequencies, is developed in Getino & Ferrándiz (1997), while the nutation series concerning the forced motion can be found in Getino & Ferrándiz (1999, 2000a).

We note that these previous studies refer to an axis-symmetric Earth model. Thus, the perturbed potential arises from the J_2 part of the geopotential coefficient which is the main contribution to the nutation series.

However, the Earth's orientation in space nowadays can be determined with great accuracy. As a direct consequence, it is necessary to consider some effects, previously disregarded, which could contribute to the nutation series and which must be taken into account in order to achieve a more complete model.

Particularly, although the Earth is almost to an axis-symmetric body, the influence of its triaxiality on the nutation is not negligible. The influence of this effect on nutations was calculated firstly by Kinoshita (1977) for the biggest terms and by Kinoshita & Souchay (1990) up to 0.005 milli arcsec More recently, Folgueira et al. (1998), Souchay et al. (1999) and Bretagnon et al. (1997) have evaluated the coefficients of nutation due to higher parts of the geopotential, obtaining the diurnal and the semidiurnal terms. Notice that all these works assume rigid Earth.

In this paper we begin with the study of the semidiurnals terms coming from the C_{22} and S_{22} geopotential coefficients by considering a non–rigid Earth model and applying the Hamiltonian formalism developed by Getino and Ferrándiz. Due to the small magnitude of these terms, in a first approximation it is enough to consider a rigid mantle, liquid core Earth model, disregarding the effect of the elasticity, whose contribution is negligible.

In Sect. 2 we summarize the formalism applied to a symmetric Earth which will be extended in Sect. 3 to take into account the effect of the triaxiality. Under the Hamiltonian framework the semidiurnal terms are obtained through the generating function, which is studied in Sect. 4. In Sect. 5 the analytical expressions of the nutations of the Andoyer and figure planes are derived. Finally, in Sect. 6 we compute the numerical values of the nutations, making a comparison with the values obtained by Souchay et al. (1999) and Bretagnon et al. (1997) for a rigid Earth model.

2. The symmetrical two-layered Earth model: An overview

In this section we give a brief description of the Hamiltonian approach to the free and forced nutations

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of a symmetrical Earth. Further information, as well as a detailed study of this problem, can be found in Getino & Ferrándiz (1997, 1999, 2000a).

2.1. Canonical variables and kinetic energy

We consider an Earth model composed of a mantle and a liquid core. As pointed out in the introduction, we can neglect the effect of the elasticity without harm, and consider the mantle as a rigid layer. Let $OXYZ$ be a non–rotating inertial frame, $OX_\mathrm{m}Y_\mathrm{m}Z_\mathrm{m}$ the frame of the principal axes of the total Earth rotating with an angular velocity $\boldsymbol{\omega}$ with respect to the inertial frame, and $OX_\mathrm{c}Y_\mathrm{c}Z_\mathrm{c}$ a core fixed frame rotating with angular velocity $\delta\boldsymbol{\omega}$ with respect to the mantle.

With an appropriate definition of the core rotation (Moritz & Mueller 1987), the angular momentum vectors of mantle and core, $\boldsymbol{L}_\mathrm{m}$ and $\boldsymbol{L}_\mathrm{c}$, can be expressed in the $OX_\mathrm{m}Y_\mathrm{m}Z_\mathrm{m}$ frame as:

$$\boldsymbol{L}_\mathrm{m} = \Pi_\mathrm{m}\,\omega, \quad \boldsymbol{L}_\mathrm{c} = \Pi_\mathrm{c}\,(\omega + \delta\omega), \tag{1}$$

where ω and $\delta\omega$ are respectively the corresponding column vectors for $\boldsymbol{\omega}$ and $\delta\boldsymbol{\omega}$ in the $OX_\mathrm{m}Y_\mathrm{m}Z_\mathrm{m}$ frame, and

$$\Pi_\mathrm{m} = \begin{pmatrix} A_\mathrm{m} & 0 & 0 \\ 0 & A_\mathrm{m} & 0 \\ 0 & 0 & C_\mathrm{m} \end{pmatrix}, \quad \Pi_\mathrm{c} = \begin{pmatrix} A_\mathrm{c} & 0 & 0 \\ 0 & A_\mathrm{c} & 0 \\ 0 & 0 & C_\mathrm{c} \end{pmatrix}. \tag{2}$$

Let us consider the total angular momentum of the Earth, $\boldsymbol{L}$, which will be:

$$\boldsymbol{L} = \boldsymbol{L}_\mathrm{m} + \boldsymbol{L}_\mathrm{c} = \Pi\,\omega + \Pi_\mathrm{c}\,\delta\omega, \tag{3}$$

where

$$\Pi = \Pi_\mathrm{m} + \Pi_\mathrm{c} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} \tag{4}$$

is the tensor of inertia of the total Earth. According to this, the components of the total angular momentum and the angular momentum of the core in the $OX_\mathrm{m}Y_\mathrm{m}Z_\mathrm{m}$ frame are:

$$\boldsymbol{L} = \begin{pmatrix} A\,\omega_1 + A_\mathrm{c}\,\delta\omega_1 \\ A\,\omega_2 + A_\mathrm{c}\,\delta\omega_2 \\ C\,\omega_3 + C_\mathrm{c}\,\delta\omega_3 \end{pmatrix}, \quad \boldsymbol{L}_\mathrm{c} = \begin{pmatrix} A_\mathrm{c}\,\omega_1 + A_\mathrm{c}\,\delta\omega_1 \\ A_\mathrm{c}\omega_2 + A_\mathrm{c}\,\delta\omega_2 \\ C_\mathrm{c}\omega_3 + C_\mathrm{c}\,\delta\omega_3 \end{pmatrix}. \tag{5}$$

Thus, the kinetic energy is written as:

$$\begin{aligned} T_0 &= \frac{1}{2}\,\boldsymbol{L}_\mathrm{m}^\mathrm{t}\,\Pi_\mathrm{m}^{-1}\,\boldsymbol{L}_\mathrm{m} + \frac{1}{2}\,\boldsymbol{L}_\mathrm{c}^\mathrm{t}\,\Pi_\mathrm{c}^{-1}\,\boldsymbol{L}_\mathrm{c} \\ &= \frac{1}{2}\,(\boldsymbol{L} - \boldsymbol{L}_\mathrm{c})^\mathrm{t}\,\Pi_\mathrm{m}^{-1}\,(\boldsymbol{L} - \boldsymbol{L}_\mathrm{c}) + \frac{1}{2}\,\boldsymbol{L}_\mathrm{c}^\mathrm{t}\,\Pi_\mathrm{c}^{-1}\,\boldsymbol{L}_\mathrm{c}. \end{aligned} \tag{6}$$

To formulate canonically the kinetic energy, we use the set of canonical variables

$$\begin{aligned} &\lambda,\, \mu, \nu,\, \Lambda,\, M,\, N, && \longrightarrow \text{ for the total Earth,} \\ &\lambda_\mathrm{c},\, \mu_\mathrm{c},\, \nu_\mathrm{c},\, \Lambda_\mathrm{c},\, M_\mathrm{c},\, N_\mathrm{c}, && \longrightarrow \text{ for the core.} \end{aligned}$$

The meaning of the angular variables is shown in Fig. 1. As for the canonical momenta, N is the OZ_m component of the total angular momentum $\boldsymbol{L}$, M is the magnitude of $\boldsymbol{L}$ and Λ is the OZ component of $\boldsymbol{L}$. Similarly for the core, N_c is the OZ_m component of the angular momentum of the core $\boldsymbol{L}_\mathrm{c}$, M_c is the magnitude of $\boldsymbol{L}_\mathrm{c}$ and Λ_c is the OZ_c component of $\boldsymbol{L}_\mathrm{c}$. Thus, by means of the auxiliary angles σ, I, σ_c and I_c, we have the relationships

$$\begin{aligned} M &= |\boldsymbol{L}| & M_\mathrm{c} &= |\boldsymbol{L}_\mathrm{c}| \\ N &= M\,\cos\sigma & N_\mathrm{c} &= M_\mathrm{c}\,\cos\sigma_\mathrm{c} \\ \Lambda &= M\,\cos I & \Lambda_\mathrm{c} &= M_\mathrm{c}\,\cos I_\mathrm{c}. \end{aligned} \tag{7}$$

Furthermore, according to Fig. 1, we can write the components of $\boldsymbol{L}$ and $\boldsymbol{L}_\mathrm{c}$ in the $OX_\mathrm{m}Y_\mathrm{m}Z_\mathrm{m}$ frame in terms of the canonical variables as follows

$$\boldsymbol{L} = \begin{pmatrix} K\sin\nu \\ K\cos\nu \\ N \end{pmatrix}, \quad \boldsymbol{L}_\mathrm{c} = \begin{pmatrix} K_\mathrm{c}\sin\nu_\mathrm{c} \\ -K_\mathrm{c}\cos\nu_\mathrm{c} \\ N_\mathrm{c} \end{pmatrix}, \tag{8}$$

where we have put

$$\begin{aligned} K &= (M^2 - N^2)^{1/2} = M\sin\sigma, \\ K_\mathrm{c} &= (M_\mathrm{c}^2 - N_\mathrm{c}^2)^{1/2} = M_\mathrm{c}\sin\sigma_\mathrm{c}\,. \end{aligned} \tag{9}$$

Finally, introducing (8) in (6) we get the canonical expression of the kinetic energy corresponding to this basic symmetrical Earth model as follows:

$$\begin{aligned} T_0 = {} & \frac{1}{2(A - A_\mathrm{c})}\left(K^2 + \frac{A}{A_\mathrm{c}}\,K_\mathrm{c}^2\right) \\ & + \frac{1}{2(C - C_\mathrm{c})}\left(N^2 - 2\,N\,N_\mathrm{c} + \frac{C}{C_\mathrm{c}}\,N_\mathrm{c}^2\right) \\ & + \frac{K\,K_\mathrm{c}}{A - A_\mathrm{c}}\,\cos(\nu + \nu_\mathrm{c}). \end{aligned} \tag{10}$$

2.2. Dissipative coupling torque

We introduce the effect of dissipative forces in the boundary mantle–core, including electromagnetic coupling and the effects of the viscosity, following the same assumptions as in Sasao et al. (1980). Thus, as a general expression we can write

$$\boldsymbol{t}_\mathrm{c} = -\boldsymbol{t}_\mathrm{m} = \begin{pmatrix} -R\,\delta\omega_1 + R'\,\delta\omega_2 \\ -R\,\delta\omega_2 - R'\,\delta\omega_1 \\ -R^*\,\delta\omega_3 \end{pmatrix}, \tag{11}$$

where R, R' and R^* are coupling constants. Note that we have written R, R', R^* instead of K, K', K^* in the original to avoid confusion in the notation. In particular, the ratio $\eta = R'/R$ is nearly zero in viscous coupling and unity in electromagnetic coupling (Sasao et al. 1980). As usual, this torque is characterized by the dimensionless complex coefficient

$$\tilde{\Gamma} = \Gamma' - i\,\Gamma. \tag{12}$$

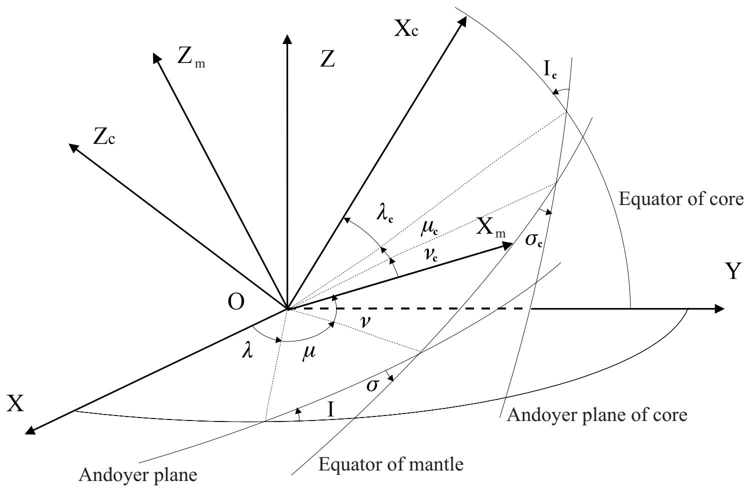

Fig. 1. Canonical variables

These new coefficients are related to the electromagnetic coupling constants R and R' by the relationships

$$\Gamma' = \frac{R'}{\Omega\, A_{\rm m}}, \quad \Gamma = \frac{R}{\Omega\, A_{\rm m}}\cdot \tag{13}$$

In the Hamiltonian framework the dissipative effects are formulated by means of the corresponding generalized forces Q_p, Q_q, whose expression, developed in Getino and Ferrándiz (1997), will be given below.

2.3. Hamiltonian for the symmetrical Earth and first order integration

According to the previous sections, the Hamiltonian of the system (to the first order) is written as

$$H = T_0 + V_0, \tag{14}$$

where V_0 is the perturbing potential coming from the J_2 geopotential coefficient, and is the same as in Kinoshita (1977) for the rigid Earth, and T_0 corresponds to the kinetic energy of an axis–symmetric rigid mantle – liquid core model, given by Eq. (10). Note that the generalized forces due to dissipative effects do not appear explicitly into the Hamiltonian (14), but they are taken into account in the corresponding equations of motion (Stiefel & Scheifele 1971). Now we perform a first order analytical integration of this Hamiltonian by using Hori's perturbation method (1966), following the same procedure as in Kinoshita (1977) for the rigid Earth. This procedure is briefly described here.

First of all, the Hamiltonian (14) is separated into an unperturbed part, H_0, corresponding to the free motion, and a perturbed part H_1, for the forced perturbations, in the form

$$H = H_0 + H_1 \rightarrow \left\{ \begin{array}{lcl} H_0 & = & T_0 \\ H_1 & = & V_0 \end{array} \right. . \tag{15}$$

Then, we carry out a canonical transformation of the initial Hamiltonian H into a new one, H^*, easier to integrate

$$H = H_0 + H_1 \rightarrow H^* = H_0^* + H_1^*, \tag{16}$$

by means of a generating function W. In the new Hamiltonian, the unperturbed part is the same as in the old one, $H_0^* = H_0$, and for the new disturbing term we take the secular part of H_1, that is to say, $H_1^* = H_{1\,\rm sec}$. We have finally

$$\begin{aligned} H_0^* \;&=\; \frac{1}{2\,A_{\rm m}} \left[K^{*2} + \frac{A}{A_{\rm c}}\,K_{\rm c}^{*2}\right] + \frac{K^*\,K_{\rm c}^*}{A_{\rm m}}\,\cos(\nu^* + \nu_{\rm c}^*) \\ &\quad + \frac{1}{2\,C_{\rm m}}\left[N^{*2} - 2\,N^*\,N_{\rm c}^* + \frac{C}{C_{\rm c}}\,N_{\rm c}^{*2}\right], \\ H_1^* \;&=\; V_{0\,\rm sec}, \end{aligned} \tag{17}$$

where expression of $V_{0\,\rm sec}$ can be found in Kinoshita (1977). Note that we have used asterisks to indicate the new variables resulting from the canonical transformation. However, in the following, these asterisks will be omitted for the sake of simplicity.

The periodic perturbations (nutations) are obtained through the generating function by means of the equations

$$\Delta q = \frac{\partial W}{\partial p^*}, \quad \Delta p = -\,\frac{\partial W}{\partial q^*} \tag{18}$$

where q stands for the angular variables (λ, μ, ν, $\lambda_{\rm c}$, $\mu_{\rm c}$, $\nu_{\rm c}$) and p for the conjugated momenta (Λ, M, N, $\Lambda_{\rm c}$, $M_{\rm c}$, $N_{\rm c}$). This generating function of the transformation is obtained through

$$W = \int (H_1 - H_1^*)\,{\rm d}t = \int H_{1\rm per}\,{\rm d}t, \tag{19}$$

where this integral is performed along the solution of the unperturbed part. This solution, corresponding to the free motion problem, is described in the next subsection.

2.4. Unperturbed solutions

The free motion problem is solved by means of the corresponding equations of motion. It is convenient to include the effect of the dissipation in this free motion problem. Thus, the equations of motion must be modified to take into account the corresponding generalized forces. These equations are of the form (Stiefel & Scheifele 1971)

$$\dot{q} = \frac{\partial T_0}{\partial p} - Q_p, \quad \dot{p} = -\frac{\partial T_0}{\partial q} + Q_q \,. \tag{20}$$

For our study, the required generalized equations of motion are (Getino & Ferrándiz 1997)

$$\begin{aligned}
&\dot{\mu} = \partial T_0/\partial M - Q_M, \quad \dot{\mu}_c = \partial T_0/\partial M_c - Q_{M_c},\\
&\dot{\nu} = \partial T_0/\partial N - Q_N, \quad \dot{\nu}_c = \partial T_0/\partial N_c - Q_{N_c},\\
&\dot{N} = -\partial T_0/\partial \nu + Q_\nu, \quad \dot{N}_c = -\partial T_0/\partial \nu_c + Q_{\nu_c},\\
&\dot{M} = -\partial T_0/\partial \mu + Q_\mu, \quad \dot{M}_c = -\partial T_0/\partial \mu_c + Q_{\mu_c},
\end{aligned} \tag{21}$$

and taking into account the expression of T_0 (10) we can write

$$\begin{aligned}
\dot{\mu} &= \frac{M}{A_m} + \frac{M}{A_m}\frac{K_c}{K}\cos(\nu + \nu_c) - Q_M,\\
\dot{\nu} &= -\frac{N}{A_m} + \frac{N - N_c}{C_m} - \frac{N}{A_m}\frac{K_c}{K}\cos(\nu + \nu_c) - Q_N\,,\\
\dot{N} &= \frac{1}{A_m} K\,K_c \sin(\nu + \nu_c) + Q_\nu,\\
\dot{M} &= Q_\mu,
\end{aligned} \tag{22}$$

for the mantle, and

$$\begin{aligned}
\dot{\mu}_c &= \frac{A}{A_c}\frac{M_c}{A_m} + \frac{M_c}{A_m}\frac{K}{K_c}\cos(\nu + \nu_c) - Q_{M_c},\\
\dot{\nu}_c &= -\frac{A}{A_c}\frac{N_c}{A_m} + \frac{1}{C_m}\left(N - \frac{C}{C_c} N_c\right)\\
&\quad -\frac{N_c}{A_m}\frac{K}{K_c}\cos(\nu + \nu_c) - Q_{N_c},\\
\dot{N}_c &= \frac{1}{A_m} K\,K_c \sin(\nu + \nu_c) + Q_{\nu_c},\\
\dot{M}_c &= Q_{\mu_c},
\end{aligned} \tag{23}$$

for the core. The generalized forces appearing in (23) are obtained in Getino & Ferrándiz (1997). The expressions of Q_i which will be used in the following are

$$\begin{aligned}
Q_{\mu_c} &= -\frac{R}{A_m}\sin\sigma_c\left[K\cos(\nu + \nu_c) + \frac{A}{A_c} K_c\right]\\
&\quad +\frac{R^*}{C_m}\cos\sigma_c\left[N - \frac{C}{C_c} N_c\right]\\
&\quad -\frac{R'}{A_m}\sin\sigma_c\, K\sin(\nu + \nu_c),\\
Q_{\nu_c} &= \frac{R^*}{C_m}\left[N - \frac{C}{C_c} N_c\right],\\
Q_{N_c} &= -\frac{R}{A_m}\frac{K}{K_c}\sin(\nu + \nu_c)\\
&\quad +\frac{R'}{A_m}\left[\frac{K}{K_c}\cos(\nu + \nu_c) + \frac{A}{A_c}\right],\\
Q_\mu &= Q_\nu = Q_M = Q_N = 0.
\end{aligned} \tag{24}$$

Now, from (22), (23) and (24) we can proceed to solve the problem. By the definition of conjugated momenta, N and N_c are respectively the third components of $\boldsymbol{L}$ and $\boldsymbol{L}_c$ (see Eqs. (5) and (8)), thus (see Getino 1995b for more details):

$$\left.\begin{aligned} N &= C\omega_3 + C_c\delta\omega_3\\ N_c &= C_c\omega_3 + C_c\delta\omega_3\end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \omega_3 &= \frac{N - N_c}{C_m},\\ \delta\omega_3 &= \frac{C\,N_c - C_c\,N}{C_m\,C_c}.\end{aligned}\right. \tag{25}$$

Now, by means of the equations of motion (22), (23) and (24), and taking into account the expression of $\delta\omega_3$ in (25) we have

$$\dot{N} - \dot{N}_c = \frac{R^*}{C_m}\left(\frac{C}{C_c} N_c - N\right) = R^*\ \delta\omega_3, \tag{26}$$

and then, from (25), we have

$$\dot{\omega}_3 = \frac{R^*}{C_m}\,\delta\omega_3\,. \tag{27}$$

On the other hand, from the second equation of (25), neglecting second order terms (product $K\,K_c$ in equation of $\dot{N}$) we obtain

$$\delta\dot{\omega}_3 = -\frac{C}{C_c}\frac{R^*}{C_m}\,\delta\omega_3, \tag{28}$$

and finally

$$\delta\omega_3 \to 0, \quad \omega_3 \to \text{constant} = \Omega\,. \tag{29}$$

Thus, taking into account Eqs. (25) and (29) we will write in the following $N = C\,\Omega$ and $N_c = C_c\,\Omega$.

With regard to the remainder equations, which provide the free frequencies of the so–called polar motion, we define the variables

$$\begin{aligned}
u &= M\sin\sigma\,(\sin\nu + i\,\cos\nu) = i\,K\,\mathrm{e}^{-i\nu},\\
v &= M_c\sin\sigma_c\,(\sin\nu_c - i\,\cos\nu_c) = -i\,K_c\,\mathrm{e}^{i\nu_c}.
\end{aligned} \tag{30}$$

With the help of Eqs. (9), (22), (23) and (24) and after some algebra, the time derivatives of variables u, v defined in Eq. (30) can be expressed in the form

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = i\,\mathrm{R}\begin{pmatrix}u\\ v\end{pmatrix}, \tag{31}$$

the matrix R being

$$\mathrm{R} = \begin{pmatrix} r_1 & r_2\\ r_3 + \Omega\,\tilde{\Gamma} & r_4 - \Omega\,\tilde{\Gamma}\,\dfrac{A}{A_c}\end{pmatrix}, \tag{32}$$

whose coefficients are

$$\begin{aligned}
r_1 &= \Omega\left(\frac{A_c}{A_m} + e\,\frac{A}{A_m}\right),\\
r_2 &= -\Omega\frac{A}{A_m}\,(1 + e)\,,\\
r_3 &= \Omega\,\frac{A_c}{A_m}\,(1 + e_c)\,,\\
r_4 &= -\Omega\,\frac{A}{A_m}\,(1 + e_c)\,,
\end{aligned} \tag{33}$$

expressed as functions of the dynamical ellipticities

$$e = \frac{C-A}{A}, \quad e_{\rm c} = \frac{C_{\rm c}-A_{\rm c}}{A_{\rm c}}, \tag{34}$$

and Γ and Γ' are the dimensionless constants of dissipation, relative to viscous and electromagnetic coupling. The free frequencies corresponding to the solutions of system (31) are as follows

$$\sigma_1 = m_1, \quad \sigma_2 = m_2 + i\,d, \tag{35}$$

with

$$\begin{aligned} m_1 &= \Omega \frac{C-A}{A_{\rm m}}, \\ m_2 &= -\Omega \left[1 + \frac{A}{A_{\rm m}}\frac{C_{\rm c}-A_{\rm c}}{A_{\rm c}} + \frac{A}{A_{\rm c}}\Gamma'\right], \\ d &= \Omega \frac{A}{A_{\rm c}}\Gamma, \end{aligned} \tag{36}$$

Another useful result, which will be used in the computation of the generating function, is derived from the equations of motion corresponding to the variables μ and ν (see Eq. (22)). From these equations we get

$$\dot\mu + \dot\nu = \Omega + \frac{M-N}{A_{\rm m}}\left(1 + \frac{K_{\rm c}}{K}\cos(\nu+\nu_{\rm c})\right). \tag{37}$$

According to definition of canonical variables (7), the second term in the r.h.s. of Eq. (37) is of the second order in σ. Therefore, due to the smallness of variables σ, $\sigma_{\rm c} \sim 10^{-6}$ rad (Nikitina 1990), neglecting this term we obtain the first order the relationship (Getino 1995b; Getino & Ferrándiz 1997)

$$\dot\mu + \dot\nu = \Omega\,. \tag{38}$$

3. Effect of the triaxiality of the Earth

This section is devoted to extending the Hamiltonian (14) corresponding to an axis–symmetric, two–layered Earth, in order to include the effect of the triaxiality. Notice that the Hamiltonian formalism has already been applied to a triaxial, two–layered model by González & Getino (1997) for the free problem, and by Getino et al. (2000), including the effect of dissipation.

In order to take into account this effect, the Hamiltonian (14) must be transformed in two ways: we include new terms in the perturbing potential corresponding to the geopotential coefficients $C_{2,2}$ and $S_{2,2}$, which characterize the triaxiality of the Earth, and the kinetic energy T is also modified.

3.1. Triaxial potential $V_{\rm Tr}$

According to Kinoshita (1977), the additional potential is

$$V_{\rm Tr} = \frac{G\,m^*}{a^{*3}}\,\frac{A-B}{4}\left(\frac{a^*}{r^*}\right)^3 P_2^2(\sin\delta)\cos 2\alpha. \tag{39}$$

Following Kinoshita, the angular part in (39) is developed in the form

$$\begin{aligned} V_{\rm Tr} &= \eta\, k' \sum_{i,\tau=\pm1}\left[\frac{3}{4}\sin^2\sigma\; B_i\;\cos(2\nu - \tau\Theta_i) \right.\\ &\quad + \sin\sigma\; C_i(\tau)\;\cos(\mu + 2\nu - \tau\Theta_i) \\ &\quad \left. + \frac{D_i(\tau)}{2}\cos(2\,\mu + 2\nu - \tau\Theta_i)\right]\cdot \end{aligned} \tag{40}$$

In expression (40) we have the coefficients

$$k' = 3\,\frac{G\,m^*}{a^{*3}}\,\frac{2C-A-B}{2}, \quad \eta = \frac{B-A}{2C-A-B}, \tag{41}$$

m^* and a^* being respectively the mass and semi–major axis concerning the perturbing body (Moon, Sun), and the functions $B_i, C_i(\tau)$ and $D_i(\tau)$ are given by:

$$\begin{aligned} B_i &= -\frac{1}{6}\left(3\cos^2 I - 1\right) A_i^{0)} - \frac{1}{2}\sin 2I\; A_i^{1)} \\ &\quad -\frac{1}{4}\sin^2 I\; A_i^{2)}, \\ C_i(\tau) &= -\frac{1}{4}\sin 2I\; A_i^{0)} + \frac{\tau}{4}\sin I\,(1+\tau\cos I)\,A_i^{2)} \\ &\quad +\frac{1}{2}(1+\tau\cos I)(-1+2\tau\cos I)\,A_i^{1)}, \\ D_i(\tau) &= -\frac{1}{2}\sin^2 I\; A_i^{0)} + \tau\sin I\,(1+\tau\cos I)\; A_i^{1)} \\ &\quad -\frac{1}{4}\,(1+\tau\cos I)^2\,A_i^{2)}, \end{aligned} \tag{42}$$

where Θ_i is a linear combination of the orbital variables of Moon and Sun, and numerical values of the coefficients $A_i^{j)}$ can be found in Kinoshita (1977), and updated in Kinoshita & Souchay (1990).

3.2. Triaxial kinetic energy $T_{\rm Tr}$

The kinetic energy corresponding to a triaxial, two–layered model has the following expression (González & Getino 1997)

$$\begin{aligned} T_{\rm Tr} &= \frac{M^2}{2}\sin^2\sigma\left(\frac{\sin^2\nu}{A_{\rm m}} + \frac{\cos^2\nu}{B_{\rm m}}\right) + \\ &\quad + \frac{M_{\rm c}^2}{2}\sin^2\sigma_{\rm c}\left(\frac{A\sin^2\nu}{A_{\rm m}\,A_{\rm c}} + \frac{B\cos^2\nu}{B_{\rm m}\,B_{\rm c}}\right) + \\ &\quad + \frac{1}{2\,C_{\rm m}}\left(N^2 - 2\,N\,N_{\rm c} + \frac{C}{C_{\rm c}}\,N_{\rm c}^2\right) + \\ &\quad + M\,M_{\rm c}\sin\sigma\,\sin\sigma_{\rm c}\left(\frac{\cos\nu\,\cos\nu_{\rm c}}{B_{\rm m}} - \frac{\sin\nu\,\sin\nu_{\rm c}}{A_{\rm m}}\right). \end{aligned} \tag{43}$$

After a little re–arrangement this term can be rewritten in a more suitable form

$$T_{\rm Tr} = T_0 + \Delta_{\rm Tr}T, \tag{44}$$

where T_0 is the kinetic energy for a symmetric Earth given by Eq. (10), and $\Delta_{\rm Tr}T$ represents the increment due to the triaxiality, which has the following expression

$$\Delta_{\rm Tr} T = \frac{A_{\rm m}-B_{\rm m}}{2\,A_{\rm m}\,B_{\rm m}}\,(M\,\sin\sigma\,\cos\nu + M_{\rm c}\,\sin\sigma_{\rm c}\,\cos\nu_{\rm c})^2 + \frac{A_{\rm c}-B_{\rm c}}{2\,A_{\rm c}\,B_{\rm c}}\,M_{\rm c}^2\,\sin^2\sigma_{\rm c}\,\cos^2\nu_{\rm c}. \quad (45)$$

3.3. Hamiltonian $H_{\rm Tr}$ and first order integration

Gathering together the previous results, the Hamiltonian for a triaxial, two–layered Earth is expressed as

$$H_{\rm Tr} = T_0 + \Delta_{\rm Tr} T + V_0 + V_{\rm Tr}. \quad (46)$$

It is interesting to break down this Hamiltonian into two parts in the form

$$H_{\rm Tr} = H_{\rm S} + \Delta_{\rm Tr} H \rightarrow \begin{cases} H_{\rm S} = T_0 + V_0, \\ \Delta_{\rm Tr} H = \Delta_{\rm Tr} T + V_{\rm Tr}, \end{cases} \quad (47)$$

where $H_{\rm S}$ is the Hamiltonian corresponding to a symmetric Earth model, whose study has been described in Sect. 2, and $\Delta_{\rm Tr} H$ encloses the terms arising from the triaxiality. Due to the small order of magnitude of $\Delta_{\rm Tr} H$ with respect to T_0, the triaxiality can be treated as a perturbation. Thus, performing a first order analytical integration described in the previous section, the periodic perturbations are given by the generating function

$$W = W_{\rm S} + W_{\rm Tr} \rightarrow \begin{cases} W_{\rm S} = \int (V_0)_{\rm per}\,{\rm d}t \\ W_{\rm Tr} = \int (\Delta_{\rm Tr} T + V_{\rm Tr})_{\rm per}\,{\rm d}t\,. \end{cases} \quad (48)$$

Periodic terms coming from $W_{\rm S}$, corresponding to the symmetric case, have been studied in Getino & Ferrándiz (2000a). We will focus our attention on $W_{\rm Tr}$ to obtain the periodic terms due to the triaxiality.

4. Generating function $W_{\rm Tr}$ and first simplifications

As described in Sect. 2 and 3, the generating function arising from the triaxiality is obtained by integrating the periodic part of $\Delta_{\rm Tr} H$ (47) over the solutions of H_0 given in Sect. 1.

In order to get $W_{\rm Tr}$, a first simplification can be performed due to the smallness of angles σ, $\sigma_{\rm c} \sim 10^{-6}$ rad. Neglecting second order terms in σ and $\sigma_{\rm c}$, in expressions (40) and (45) for $V_{\rm T_r}$ and $\Delta_{\rm T_r} T$, the generating function reduces to

$$W_{\rm Tr} = \eta\,k' \sum_{i,\tau=\pm1} \int \Big[\sin\sigma\; C_i(\tau)\,\cos(\mu + 2\nu - \tau\Theta_i) + \frac{D_i(\tau)}{2}\,\cos(2\,\mu + 2\nu - \tau\Theta_i)\Big]\,{\rm d}t. \quad (49)$$

To carry out this integration, we follow the same technique developed in Getino & Ferrándiz (1997), which is summarized here.

4.1. Auxiliary integrals

Let us begin by computing two integrals which will be very useful in the development of this technique. These integrals are written as

$$I_3 = \int M\,\sin\sigma\,\cos(\mu + 2\nu - \tau\,\Theta_i)\,{\rm d}t,$$
$$I_4 = \int M_{\rm c}\,\sin\sigma_{\rm c}\,\cos(\mu + \nu - \nu_{\rm c} - \tau\,\Theta_i)\,{\rm d}t. \quad (50)$$

Let P and Q be new integrals whose imaginary parts are respectively I_3 and I_4, that is, $I_3 = {\rm Im}\{P\}$, $I_4 = {\rm Im}\{Q\}$, then defined by

$$P = \int i\,M\,\sin\sigma\,{\rm e}^{i\,\nu}\,{\rm e}^{i\,h}\,{\rm d}t,$$
$$Q = \int i\,M_{\rm c}\,\sin\sigma_{\rm c}\,{\rm e}^{-i\,\nu_{\rm c}}\,{\rm e}^{i\,h}\,{\rm d}t, \quad (51)$$

where we have introduced the notation

$$h = \mu + \nu - \tau\,\Theta_i. \quad (52)$$

On the other hand, the variables u, v in Eq. (30) can be expressed as

$$u = i\,M\,\sin\sigma\,{\rm e}^{-i\,\nu}, \quad v = -i\,M_{\rm c}\,\sin\sigma_{\rm c}\,{\rm e}^{i\,\nu_{\rm c}}. \quad (53)$$

From Eqs. (51) and (53) the new integrals are written as

$$P = -\int u^*\,{\rm e}^{i\,h}\,{\rm d}t, \quad Q = \int v^*\,{\rm e}^{i\,h}\,{\rm d}t, \quad (54)$$

where u^*, v^* represent respectively the complex conjugates of u, v.

Now let us pay attention to the system of equations for the free motion given by Eq. (31). This system can be transformed into

$$\begin{pmatrix} \int \dot{u}^*\,{\rm e}^{i\,h}\,{\rm d}t \\ \int \dot{v}^*\,{\rm e}^{i\,h}\,{\rm d}t \end{pmatrix} = -i\,{\rm R}^* \begin{pmatrix} \int u^*\,{\rm e}^{i\,h}\,{\rm d}t \\ \int v^*\,{\rm e}^{i\,h}\,{\rm d}t \end{pmatrix}. \quad (55)$$

The l.h.s. of Eq. (55) can be integrated by parts, and we obtain

$$\begin{pmatrix} \int \dot{u}^*\,{\rm e}^{i\,h}\,{\rm d}t \\ \int \dot{v}^*\,{\rm e}^{i\,h}\,{\rm d}t \end{pmatrix} = \begin{pmatrix} u^*\,{\rm e}^{i\,h} \\ v^*\,{\rm e}^{i\,h} \end{pmatrix} - i\,n_h \begin{pmatrix} \int u^*\,{\rm e}^{i\,h}\,{\rm d}t \\ \int v^*\,{\rm e}^{i\,h}\,{\rm d}t \end{pmatrix}, \quad (56)$$

where, taking into account Eq. (38),

$$n_h = \frac{{\rm d}h}{{\rm d}t} = \dot{\mu} + \dot{\nu} - \tau\,\dot{\Theta}_i = \Omega - \tau\,n_i\,. \quad (57)$$

From Eqs. (51), (55) and (56) we get

$$\begin{pmatrix} P \\ -Q \end{pmatrix} = \frac{({\rm R}^* - n_h\,1)^\alpha}{|{\rm R}^* - n_h\,1|} \begin{pmatrix} -\,i\,u^*\,{\rm e}^{i\,h} \\ -\,iv^*\,{\rm e}^{i\,h} \end{pmatrix}, \quad (58)$$

where the superscript α stands for the adjoint matrix, and 1 is the unit matrix. As the eigenvalues of matrix $\rm R^*$ are the complex conjugates of those of matrix R, being these ones the free frequencies σ_1 and σ_2 given by Eqs. (35) and

(36), taking into account the expression of matrix R in Eqs. (32) and (33), we have that

$$
\begin{aligned}
|\mathrm{R}^* - n_h\,1| &= (n_h-\sigma_1^*)\,(n_h-\sigma_2^*),\\
(\mathrm{R}^* - n_h\,1)^\alpha &= \begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\\
&= \begin{pmatrix} -n_h + r_4 - \Omega\,\dfrac{A}{A_\mathrm{c}}\,\tilde\Gamma^* & -r_2\\ -r_3 - \Omega\,\tilde\Gamma^* & -n_h + r_1\end{pmatrix},
\end{aligned}
\tag{59}
$$

then, taking into account Eq. (53), the integrals P and Q are given by

$$
\begin{aligned}
P &= -\,M\sin\sigma\,\mathrm{e}^{i(h+\nu)}\,\frac{c_{11}}{(n_h-\sigma_1^*)(n_h-\sigma_2^*)}\\
&\quad + M_\mathrm{c}\sin\sigma_\mathrm{c}\,\mathrm{e}^{i(h-\nu_\mathrm{c})}\,\frac{c_{12}}{(n_h-\sigma_1^*)(n_h-\sigma_2^*)},\\
Q &= -\,M\sin\sigma\,\mathrm{e}^{i(h+\nu)}\,\frac{c_{21}}{(n_h-\sigma_1^*)(n_h-\sigma_2^*)}\\
&\quad + M_\mathrm{c}\sin\sigma_\mathrm{c}\,\mathrm{e}^{i(h-\nu_\mathrm{c})}\,\frac{c_{22}}{(n_h-\sigma_1^*)(n_h-\sigma_2^*)},
\end{aligned}
\tag{60}
$$

Finally, taking the imaginary parts in Eq. (60), and neglecting second order terms in ellipticities and dissipation, expressions of integrals I_3, I_4 are obtained in the form

$$
\begin{aligned}
I_3 &= M\,\sin\sigma\,\left[\hat F_1^a\sin(h+\nu) + \hat F_1^b\cos(h+\nu)\right]\\
&\quad + M_\mathrm{c}\,\sin\sigma_\mathrm{c}\,\left[\hat F_2^a\sin(h-\nu_\mathrm{c}) + \hat F_2^b\cos(h-\nu_\mathrm{c})\right],\\
I_4 &= M\,\sin\sigma\,\left[\hat G_1^a\sin(h+\nu) + \hat G_1^b\cos(h+\nu)\right]\\
&\quad + M_\mathrm{c}\,\sin\sigma_\mathrm{c}\,\left[\hat G_2^a\sin(h-\nu_\mathrm{c}) + \hat G_2^b\cos(h-\nu_\mathrm{c})\right],
\end{aligned}
\tag{61}
$$

where,with the help of the notation

$$
\begin{aligned}
\hat f_1 &= n_h - m_1 = \Omega - \tau\,n_i - m_1,\\
\hat f_2 &= n_h - m_2 = \Omega - \tau\,n_i - m_2,
\end{aligned}
\tag{62}
$$

the following functions have been defined:

$$
\begin{aligned}
\hat F_1^a &= \frac{n_h - r_4 + \Omega\,\frac{A}{A_\mathrm{c}}\,\Gamma'}{\hat f_1\,\hat f_2},\\
\hat F_1^b &= -\,\Omega\,\Gamma\,\frac{A}{A_\mathrm{c}}\,\frac{n_h - r_4 - \hat f_2}{\hat f_1\,\hat f_2^2},\\
\hat G_1^a &= \frac{-\,r_3 - \Omega\,\Gamma'}{\hat f_1\,\hat f_2},\\
\hat G_1^b &= -\,\Omega\,\Gamma\,\frac{\hat f_2 - \frac{A}{A_\mathrm{c}}\,r_3}{\hat f_1\,\hat f_2^2},\\
\hat F_2^a &= \frac{-\,r_2}{\hat f_1\,\hat f_2},\\
\hat F_2^b &= \Omega\,\Gamma\,\frac{A}{A_\mathrm{c}}\,\frac{r_2}{\hat f_1\,\hat f_2^2},\\
\hat G_2^a &= \frac{n_h - r_1}{\hat f_1\,\hat f_2},\\
\hat G_2^b &= -\,\Omega\,\Gamma\,\frac{A}{A_\mathrm{c}}\,\frac{n_h - r_1}{\hat f_1\,\hat f_2^2}.
\end{aligned}
\tag{63}
$$

4.2. Generating function

From expressions (49) and (40), the term of the generating function due to the triaxiality can be divided into two parts

$$
W_\mathrm{Tr} = \int V_\mathrm{Tr}\,\mathrm{d}t = W_\mathrm{C} + W_\mathrm{D},
\tag{64}
$$

with

$$
\begin{aligned}
W_\mathrm{C} &= \eta\,k'\sum_{i,\tau=\pm1} C_i(\tau)\int \sin\sigma\,\cos(\mu+2\nu-\tau\Theta_i)\,\mathrm{d}t,\\
W_\mathrm{D} &= \eta\,k'\sum_{i,\tau=\pm1} \frac{D_i(\tau)}{2}\int \cos(2\mu+2\nu-\tau\Theta_i)\,\mathrm{d}t.
\end{aligned}
\tag{65}
$$

The term W_D is the same as in Kinoshita (1977) for the rigid Earth. Taking into account Eq. (38), it is straightforward to obtain

$$
W_\mathrm{D} = \frac{\eta\,k'}{2}\sum_{i,\tau=\pm1}\frac{D_i(\tau)}{2\Omega-\tau n_i}\sin(2\mu+2\nu-\tau\Theta_i),
\tag{66}
$$

were $n_i = \dot\Theta_i$. Now, by means of integral I_3 (61), the term W_C is expressed as

$$
\begin{aligned}
W_\mathrm{C} &= \eta\,k'\sum_{i,\tau=\pm1} C_i(\tau)\left[\sin\sigma\,\left(\hat F_1^a\,\sin a_1 + \hat F_1^b\,\cos a_1\right)\right.\\
&\quad \left. + \frac{M_\mathrm{c}}{M}\,\sin\sigma_\mathrm{c}\,\left(\hat F_2^a\,\sin a_2 + \hat F_2^b\,\cos a_2\right)\right],
\end{aligned}
\tag{67}
$$

with the arguments

$$
a_1 = \mu + 2\nu - \tau\Theta_i,\; a_2 = \mu + \nu - \nu_\mathrm{c} - \tau\Theta_i.
\tag{68}
$$

5. Forced nutations

The periodic perturbations, forced nutations, are obtained through the generating function by means of the well–known relationships

$$
\begin{aligned}
\Delta_\mathrm{Tr}(\mu,\nu,\lambda) &= \frac{\partial W_\mathrm{Tr}}{\partial(M,N,\Lambda)},\\
\Delta_\mathrm{Tr}(M,N,\Lambda) &= \frac{-\,\partial W_\mathrm{Tr}}{\partial(\mu,\nu,\lambda)}.
\end{aligned}
\tag{69}
$$

Next we compute the periodic perturbations of the fundamental planes, that is to say, the plane perpendicular to the angular momentum vector (or Andoyer plane), and the plane perpendicular to the figure axis of the Earth (figure plane or equatorial plane).

5.1. Nutations of the Andoyer plane

The longitude of the node and the inclination of this plane are given respectively by λ and $I = \cos^{-1}(\Lambda/M)$. The nutations corresponding to these variables, known as Poisson terms, are obtained through the equations (Kinoshita 1977)

$$\Delta_{\rm Tr}\lambda = \frac{-1}{M\sin I}\frac{\partial W_{\rm Tr}}{\partial I},$$

$$\Delta_{\rm Tr} I = \frac{1}{M\sin I}\left(\frac{\partial W_{\rm Tr}}{\partial\lambda} - \cos I\,\frac{\partial W_{\rm Tr}}{\partial\mu}\right). \quad (70)$$

Neglecting second order terms in small parameters, the contribution of these nutations comes from the term $W_{\rm D}$ of the generating function, which corresponds to the rigid perturbation. Using the angles $\psi = -\lambda$, $\varepsilon = -I$ and taking into account the relationships between functions $C_i(\tau)$ and $D_i(\tau)$ (see Kinoshita 1977), the Poisson terms at the first order are as follows

$$\Delta^{\rm D}_{\rm Tr}\psi = -\eta\,\frac{k_0}{\sin\varepsilon}\sum_{i,\tau=\pm1}\frac{C_i(\tau)}{2\Omega-\tau\,n_i}\sin\left(2\phi'-\tau\Theta_i\right),$$

$$\Delta^{\rm D}_{\rm Tr}\varepsilon = \eta\,k_0\sum_{i,\tau=\pm1}\frac{C_i(\tau)}{2\Omega-\tau\,n_i}\cos\left(2\phi'-\tau\Theta_i\right), \quad (71)$$

with

$$k_0 = \frac{k'_0}{M}\simeq\frac{3\,G\,m^*}{{a^*}^3\Omega}\,\frac{C-A}{A},$$

$$\phi' = \mu+\nu. \quad (72)$$

Expressions (71) are the same as those of Kinoshita (1977) for the rigid Earth. That is, the presence of the liquid core and the effects of dissipation do not influence the nutations of the Andoyer plane, which is consistent with the fact that the motion of the angular momentum axis does not depend on the internal constitution of the Earth (Moritz & Mueller 1987).

5.2. Nutations of the figure plane

The longitude of the node, $\lambda_{\rm f}$, and the inclination $I_{\rm f}$ of this plane are given to the first order by (Kinoshita 1977)

$$\lambda_{\rm f} = \lambda + \frac{\sigma}{\sin I}\sin\mu, \qquad I_{\rm f} = I + \sigma\,\cos\mu\,. \quad (73)$$

Following Kinoshita, the periodic perturbations of the increments $\lambda_{\rm f}-\lambda$, $I_{\rm f}-I$, called Oppolzer terms, are given to the first order by

$$\Delta_{\rm Tr}(\lambda_{\rm f}-\lambda) = \frac{1}{\sin I}\,\frac{1}{M\sin\sigma}\times\left[\sin\mu\left(\frac{\partial W_{\rm Tr}}{\partial\nu} - \frac{\partial W_{\rm Tr}}{\partial\mu}\right) + \sigma\cos\mu\frac{\partial W_{\rm Tr}}{\partial\sigma}\right],$$

$$\Delta_{\rm Tr}(I_{\rm f}-I) = \frac{1}{M\sin\sigma}\times\left[\cos\mu\left(\frac{\partial W_{\rm Tr}}{\partial\nu} - \frac{\partial W_{\rm Tr}}{\partial\mu}\right) - \sigma\sin\mu\frac{\partial W_{\rm Tr}}{\partial\sigma}\right]. \quad (74)$$

Neglecting second order, the contribution to the Oppolzer terms comes from $W_{\rm C}$. From Eq. (67), after some algebra we finally obtain, in terms of the angles $\psi_{\rm f} = -\lambda_{\rm f}$, $\varepsilon_{\rm f} = -I_{\rm f}$, the following expressions

$$\Delta_{\rm Tr}(\psi_{\rm f}-\psi) = \frac{\eta\,k_0}{\sin\varepsilon}\sum_{i,\tau=\pm1}C_i(\tau)\times\left[\hat F^a_1\,\sin\left(2\phi'-\tau\Theta_i\right) + \hat F^b_1\,\cos\left(2\phi'-\tau\Theta_i\right)\right],$$

$$\Delta_{\rm Tr}(\varepsilon_{\rm f}-\varepsilon) = -\eta\,k_0\sum_{i,\tau=\pm1}C_i(\tau)\times\left[\hat F^a_1\,\cos\left(2\phi'-\tau\Theta_i\right) - \hat F^b_1\,\sin\left(2\phi'-\tau\Theta_i\right)\right]. \quad (75)$$

5.3. Comparison with the rigid Earth results of Kinoshita (1977)

As we have pointed out, Poisson terms (71) are the same as in the rigid case. In order to compare Oppolzer terms (75) with those of Kinoshita (1977) for the rigid case, we must translate his results to our notation. Namely, the Oppolzer terms, due to the triaxiality of the Earth, of Kinoshita (1977), can be written as

$$\Delta_{\rm Tr}(\psi_{\rm f}-\psi))_{\rm rigid} = \frac{\eta\,k_0}{\sin\varepsilon}\times\sum_{i,\tau=\pm1}\frac{C_i(\tau)\;\sin\left(2\phi'-\tau\Theta_i\right)}{\dot\mu)_{\rm rigid}+2\,\dot\nu)_{\rm rigid}-\tau n_i},$$

$$\Delta_{\rm Tr}(\varepsilon_f-\varepsilon))_{\rm rigid} = -\eta\,k_0\times\sum_{i,\tau=\pm1}\frac{C_i(\tau)\;\cos\left(2\phi'-\tau\Theta_i\right)}{\dot\mu)_{\rm rigid}+2\,\dot\nu)_{\rm rigid}-\tau n_i}, \quad (76)$$

where $\dot\mu)_{\rm rigid}$ and $\dot\nu)_{\rm rigid}$ are given by (see Kinoshita 1977)

$$\dot\mu)_{\rm rigid}\simeq\frac{C}{A}\Omega,\;\dot\nu)_{\rm rigid}\simeq-\frac{C-A}{A}\Omega. \quad (77)$$

From Eqs. (75) and (76) it follows that Oppolzer terms are substantially different for the rigid and non-rigid cases. On the one hand, in the non-rigid case there are in–phase and out–of–phase terms. Out–of–phase terms come from the function $\hat F^b_1$, that, according to Eq. (63), is produced by the dissipation at the CMB (through the coefficients Γ). On the other hand, amplitudes of in–phase terms, given by $\hat{F}_1^a$, are not the same as in the rigid case.

We can check that our non-rigid model coincides with that of Kinoshita (1977) when removing the non-rigidity of the Earth. With this aim, we must take into account that to convert our model into the rigid one the next steps must be followed. Firstly, in the rigid Earth there is no dissipation of electromagnetic or viscous origin, so we must take $\tilde{\Gamma} = 0$. Secondly, the rigid Earth is composed of only one layer, the mantle, therefore we must take $e_{\rm c} = 0$ and $A_{\rm m} = A$. Bearing these considerations in mind we arrive at

$$\left.\hat{F}_1^a\right)_{\rm rigid} = \left(\Omega - \tau n_i - \Omega\frac{C-A}{A}\right)^{-1},$$

$$\left.\hat{F}_b^a\right)_{\rm rigid} = 0, \tag{78}$$

that, with the help of Eq. (77), can be re–written as

$$\left.\hat{F}_1^a\right)_{\rm rigid} = \left(\left.\dot{\mu}\right)_{\rm rigid} + 2\left.\dot{\nu}\right)_{\rm rigid} - \tau n_i\right)^{-1},$$

$$\left.\hat{F}_b^a\right)_{\rm rigid} = 0. \tag{79}$$

Substituting these functions into Eqs. (75) we recover the Oppolzer terms given by Kinoshita (1977), Eqs. (76).

In order to gain more insight into the difference between the rigid and non–rigid situations, let us introduce the ratio between the in-phase amplitude for the rigid and non-rigid nutations corresponding to the same argument $(2\phi' - \tau\Theta_i)$. Taking into account Eqs. (36), (62), (63) and (79) we obtain

$$\left.\frac{\hat{F}_1^a}{\left.\hat{F}_1^a\right)_{\rm rigid}}\right|_{2\phi'-\tau\Theta_i} = \frac{(1 - e - \tau n_i/\Omega)}{(1 - A/A_{\rm m}e - \tau n_i/\Omega)} \times\frac{[1 + A/A_{\rm m}\,(1+e_{\rm c}) - \tau n_i/\Omega]}{(2 + A/A_{\rm m}e_{\rm c} - \tau n_i/\Omega)}. \tag{80}$$

This equation shows that the ratio between the in-phase amplitudes for the non-rigid and rigid models is a function of some Earth parameters and of the frequency n_i corresponding to each term. It is important to note that due to the functional dependence in Eq. (80) and the values of n_i, there is no possibility of having small divisors, as happens to the analogous ratio for the usual long–period terms coming from J_2. That is, there is no noticeable resonant effect. The former ratio varies very slightly in the band of the disturbing frequencies n_i and is nearly constant. A straightforward Taylor expansion of this ratio with respect to the ellipticities and the frequencies n_i provides that its value deviates from 1 in about $A_{\rm c}/2\,A_{\rm m}$. This last coefficient is a common factor of terms in n_i and the ellipticities, that have therefore a small effect.

The ratio given by Eq. (80) is not a "transfer function" (the ratio of the whole amplitudes of non–rigid and rigid nutations for each period, gathering both Poisson and Oppolzer terms). The true expression of the transfer function can easily be computed from Eqs. (83), (76) and (71). This transfer function is frequency–dependent from the mathematical perspective, although in practice is nearly equal to $1 + A_{\rm c}/A_{\rm m} \simeq 1.123$ in the band of the relevant disturbing frequencies.

5.4. Reference to Greenwich prime meridian

Note that the angle $\phi' = \mu + \nu$ refers to the principal axis of the Earth corresponding to the minimum moment of inertia, according to the definition of the Andoyer angle ν (see Kinoshita 1977; Getino 1995a). The angle ϕ of sidereal rotation of the Earth, referred to the Greenwich prime meridian, is obtained by a phase shift in angle ϕ' (Bretagnon et al. 1997). Thus, we can write a relationship of the form

$$2\phi = 2\phi' + g. \tag{81}$$

This phase shift is related to the geopotential coefficients J_2, $C_{2,2}$ and $S_{2,2}$ by expressions

$$\eta\,\cos g = \frac{2\,C_{2,2}}{J_2},\quad \eta\,\sin g = \frac{2\,S_{2,2}}{J_2}. \tag{82}$$

Finally, introducing Eq. (82) in expressions (71) and (75) we get the following expressions for the nutations of the figure axis

$$\begin{aligned}\Delta_{\rm Tr}\varepsilon_{\rm f} = {} & \frac{2k_0}{J_2}\sum_{i,\tau=\pm1} C_i(\tau)\left[\frac{1}{2\Omega - \tau n_i} - \hat{F}_1^a\right] \\ & \times\left[C_{2,2}\cos(2\phi - \tau\Theta_i) - S_{2,2}\sin(2\phi - \tau\Theta_i)\right] \\ & + \frac{2k_0}{J_2}\sum_{i,\tau=\pm1} C_i(\tau)\,\hat{F}_1^b \\ & \times\left[C_{2,2}\sin(2\phi - \tau\Theta_i) + S_{2,2}\cos(2\phi - \tau\Theta_i)\right],\end{aligned}$$

$$\begin{aligned}\Delta_{\rm Tr}\psi_{\rm f} = {} & \frac{-2k_0}{J_2\,\sin\varepsilon}\sum_{i,\tau=\pm1} C_i(\tau)\left[\frac{1}{2\Omega - \tau n_i} - \hat{F}_1^a\right] \\ & \times\left[C_{2,2}\sin(2\phi - \tau\Theta_i) + S_{2,2}\cos(2\phi - \tau\Theta_i)\right] \\ & - \frac{2k_0}{J_2\,\sin\varepsilon}\sum_{i,\tau=\pm1} C_i(\tau)\,\hat{F}_1^b \\ & \times\left[C_{2,2}\cos(2\phi - \tau\Theta_i) - S_{2,2}\sin(2\phi - \tau\Theta_i)\right].\end{aligned} \tag{83}$$

6. Nutation series

We proceed in this section to a numerical computation of the forced nutations given by Eqs. (71) for the angular momentum axis and (83) for the figure axis. First of all we take the following numerical values from the IERS standards (McCarthy 1996)

Parameter	IERS standards
Ω	$7.292115\ 10^{-5}$ rads^{-1}
ε	$84381.''412$
J_2	$1082.6359\ 10^{-6}$
$C_{2,2}$	$1.574410\ 10^{-6}$
$S_{2,2}$	$-0.903757\ 10^{-6}$

The nutations depend on a set of parameters characterizing the Earth model used, through coefficient r_4, Γ and functions $\hat{f}_1, \hat{f}_2$ appearing in definition of functions $F_1^{a,b}$, as well as the coefficients $k_{0\mathrm{M}}$ and $k_{0\mathrm{S}}$ of the perturbing potential. Analytical formulae of the forced nutations (71) and (83) can be expressed as functions of the following set of basic Earth parameters (BEP)

$$BEP = \{P_{\mathrm{CW}},\, P_{\mathrm{FCN}},\, A_{\mathrm{c}}/A_{\mathrm{m}},\, \Gamma,\, k_0)_{\mathrm{M}},\, k_0)_{\mathrm{S}}\}\,,$$

where P_{CW} and P_{FCN} are respectively the periods corresponding to the free frequencies m_1 and m_2 (36), that is to say,

$$P_{\mathrm{CW}} = \left[e\,\frac{A}{A_{\mathrm{m}}}\right]^{-1},$$

$$P_{\mathrm{FCN}} = \left[\frac{A}{A_{\mathrm{m}}}\,e_{\mathrm{c}} + \frac{A}{A_{\mathrm{c}}}\,\Gamma'\right]^{-1}, \qquad (84)$$

$A_{\mathrm{c}}/A_{\mathrm{m}}$ is the ratio between principal moments of core and mantle, Γ is the coefficient of the dissipation and $k_0)_{\mathrm{M}}$, $k_0)_{\mathrm{S}}$ are the coefficients of the perturbing potential (72) for the Moon and Sun. Finally, by means of Eqs. (33), (36), (57), (62), (63) and (84) we get the expressions

$$\hat{F}_1^a = \frac{2 + A_{\mathrm{c}}/A_{\mathrm{m}} + P_{\mathrm{FCN}}^{-1} - \tau\, n_i/\Omega}{\Omega\left(1 - P_{\mathrm{CW}}^{-1} - \tau\, n_i/\Omega\right)\left(2 + P_{\mathrm{FCN}}^{-1} - \tau\, n_i/\Omega\right)},$$

$$\hat{F}_1^b = \frac{-\,\Gamma\,(1 + A_{\mathrm{c}}/A_{\mathrm{m}})}{\Omega\left(1 - P_{\mathrm{CW}}^{-1} - \tau\, n_i/\Omega\right)\left(2 + P_{\mathrm{FCN}}^{-1} - \tau\, n_i/\Omega\right)}\,. \qquad (85)$$

Values of BEP, taken from Getino and Ferrándiz (2000b), are the following

Parameter	Getino and Ferrándiz
P_{CW}	400.7 days
P_{FCN}	432.94 days
$A_{\mathrm{c}}/A_{\mathrm{m}}$	0.123234
Γ	$4.1\ 10^{-6}$
$k_0)_{\mathrm{M}}$	$7567.''870647$/Jcy
$k_0)_{\mathrm{S}}$	$3474.''613747$/Jcy

With these parameters the amplitudes of the nutation series can be evaluated. First of all, we show in Table 1 the amplitudes corresponding to the dissipative terms in obliquity (depending on function $\hat{F}_1^b$), where only the main terms are listed. As expected, these values are very small, so that the effect of the dissipation on the semi–diurnal terms can be disregarded in the future.

Main amplitudes of non–dissipative terms are listed in Tables 2 (obliquity) and 3 (longitude), where we have included the amplitudes for angular momentum axis (Poisson terms) and figure axis. Finally in Tables 4 and 5 we compare our results for the amplitudes of the figure axis with those of Bretagnon et al. (1997) and Souchay et al. (1999) for the rigid Earth.

Table 1. Semi–diurnal terms: nutations in obliquity. Dissipative terms (Unit = μas)

Argument						Figure Axis	
ϕ	l_{M}	l_{S}	F	D	Ω	C_{22}(sin)	S_{22}(cos)
2	0	0	0	0	0	.029	−.016
2	0	0	0	0	−1	.003	−.002
2	0	0	−2	2	−2	−.009	.005
2	−1	0	0	0	0	.001	−.001
2	0	0	−2	0	−2	−.024	.013
2	0	0	−2	0	−1	−.004	.002
2	−1	0	−2	0	−2	−.004	.002

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Table 2. Semi–diurnal terms: nutations in obliquity (Unit = μas)

Argument						Period	Angular Momentum		Figure Axis		Alias
ϕ	$l_{\rm M}$	$l_{\rm S}$	F	D	Ω	(days)	C_{22}(cos)	S_{22}(sin)	C_{22}(cos)	S_{22}(sin)	period
2	0	0	0	0	0	.498634	12.670	7.273	−14.297	−8.207	∞
2	0	0	0	0	−1	.498598	1.717	.986	−1.938	−1.112	−6798.36
2	0	0	−2	2	−2	.500000	−4.202	−2.412	4.768	2.737	182.62
2	0	−1	−2	2	−2	.500685	−.246	−.141	.280	.161	121.75
2	−1	0	0	0	0	.507826	.721	.414	−.844	−.485	27.55
2	1	0	−2	0	-2	.507985	.259	.148	−.304	−.174	27.09
2	0	0	−2	0	−2	.517526	−9.351	−5.367	11.385	6.535	13.66
2	0	0	−2	0	−1	.517569	−1.763	−1.012	2.147	1.232	13.63
2	1	0	−2	−2	−2	.526074	−.345	−.198	.435	.250	9.56
2	−1	0	−2	0	−2	.527441	−1.824	−1.047	2.312	1.327	9.13
2	−1	0	−2	0	−1	.527474	−.344	−.197	.436	.250	9.12
2	0	0	−2	−2	−2	.536299	−.296	−.170	.389	.223	7.10
2	−2	0	−2	0	−2	.537720	−.246	−.141	.325	.186	6.86
2	0	0	0	0	1	.498671	−.251	−.144	.283	.162	6798.36
2	0	0	2	−2	2	.497277	.179	.103	−.201	−.115	−182.62
2	1	0	0	0	0	.489770	.695	.399	−.757	−.435	−27.55
2	0	0	2	0	2	.481074	.374	.214	−.393	−.225	−13.66
2	0	0	2	0	1	.481036	.239	.137	−.251	−.144	−13.63

Table 3. Semi–diurnal terms: nutations in longitude (Unit = μas)

Argument						Period	Angular Momentum		Figure Axis		Alias
ϕ	$l_{\rm M}$	$l_{\rm S}$	F	D	Ω	(days)	C_{22}(sin)	S_{22}(cos)	C_{22}(sin)	S_{22}(cos)	period
2	0	0	0	0	0	.498634	−31.853	18.285	35.944	−20.632	∞
2	0	0	0	0	−1	.498598	−4.318	2.478	4.872	−2.796	−6798.36
2	0	0	−2	2	−2	.500000	10.565	−6.064	−11.987	6.881	182.62
2	0	−1	−2	2	−2	.500685	.620	−.355	−.705	.404	121.75
2	−1	0	0	0	0	.507826	−1.814	1.041	2.124	−1.219	27.55
2	1	0	−2	0	−2	.507985	−.652	.374	.764	−.438	27.09
2	0	0	−2	0	−2	.517526	23.509	−13.494	−28.622	16.430	13.66
2	0	0	−2	0	−1	.517569	4.433	−2.545	−5.399	3.099	13.63
2	1	0	−2	−2	−2	.526074	.869	−.498	−1.095	.628	9.56
2	−1	0	−2	0	−2	.527441	4.587	−2.633	−5.812	3.336	9.13
2	−1	0	−2	0	−1	.527474	.864	−.496	−1.096	.629	9.12
2	0	0	−2	−2	−2	.536299	.744	−.427	−.978	.561	7.10
2	−2	0	−2	0	−2	.537720	.618	−.355	−.817	.469	6.86
2	0	0	0	0	1	.498671	.631	−.362	−.712	.408	6798.36
2	0	0	2	−2	2	.497277	−.452	.259	.507	−.291	−182.62
2	1	0	0	0	0	.489770	−1.749	1.004	1.905	−1.093	−27.55
2	0	0	2	0	2	.481074	−.940	.539	.988	−.567	−13.66
2	0	0	2	0	1	.481036	−.602	.345	.633	−.363	−13.63

Table 4. Comparison with rigid Earth theories. Figure axis. Obliquity (Unit = μas)

Argument						Period	Bretagnon et al.		Souchay et al.		Authors	
ϕ	$l_{\rm M}$	$l_{\rm S}$	F	D	Ω	(days)	$\Delta\varepsilon_{\rm f}$ (cos)	$\Delta\varepsilon_{\rm f}$ (sin)	$\Delta\varepsilon_{\rm f}$ (cos)	$\Delta\varepsilon_{\rm f}$ (sin)	$\Delta\varepsilon_{\rm f}$ (cos)	$\Delta\varepsilon_{\rm f}$ (sin)
2	−1	0	−2	0	−2	.527441	2.05	1.17	2.06	1.18	2.312	1.327
2	0	0	−2	0	−1	.517569	1.90	1.09	1.88	1.08	2.147	1.232
2	0	0	−2	0	−2	.517526	10.08	5.79	9.89	5.68	11.385	6.535
2	−1	0	0	0	0	.507826	−0.74	−0.43	−0.72	−0.42	−.844	−.485
2	0	0	−2	2	−2	.500000	4.22	2.42	4.12	2.36	4.768	2.737
2	0	0	0	0	0	.498634	−12.67	−7.27	−12.40	−7.12	−14.297	−8.207
2	0	0	0	0	−1	.498598	−1.72	−0.99	−1.66	−0.95	−1.938	−1.112

Table 5. Comparison with rigid Earth theories. Figure axis. Longitude (Unit = μas)

Argument						Period	Bretagnon et al.		Souchay et al		Authors	
ϕ	l_M	l_S	F	D	Ω	(days)	$\Delta\psi_f$(sin)	$\Delta\psi_f$(cos)	$\Delta\psi_f$(sin)	$\Delta\psi_f$(cos)	$\Delta\psi_f$(sin)	$\Delta\psi_f$(cos)
2	−1	0	−2	0	−2	.527441	−5.15	2.96	−5.10	2.93	−5.812	3.336
2	0	0	−2	0	1	.517569	−4.32	2.48	−4.71	2.70	−5.399	3.099
2	0	0	−2	0	−2	.517526	−25.36	14.55	−25.53	14.65	−28.622	16.430
2	−1	0	0	0	0	.507826	1.88	−1.08	1.81	−1.04	2.124	−1.219
2	0	0	−2	2	−2	.500000	−10.63	6.10	−10.45	6.00	−11.987	6.881
2	0	0	0	0	0	.498634	31.85	−18.28	31.82	−18.24	35.944	−20.632
2	0	0	0	0	−1	.498598	4.32	−2.48	4.28	−2.45	4.872	−2.796

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