A&A 369, 677–689 (2001)
DOI: 10.1051/0004-6361:20010141

Astronomy & Astrophysics

Hamiltonian formulation of the secular theory for Trojan-type motion

M. H. M. Morais*

Observatoire de la Côte d'Azur, BP 4229, 06304 Nice Cedex 4, France

Received 19 October 2000 / Accepted 12 January 2001

Abstract. We re-derive the secular theory for Trojan-type motion from Morais (1999) using a Hamiltonian formulation and show how this methodology allows us to include the effect of an oblate central mass and the secular perturbations from additional bodies in a rigorous way. As an application of this work we locate secular resonances inside the co-orbital regions of the uranian satellites and the planets, and show that these are in good agreement with the behaviour observed in numerical integrations.

Key words. celestial mechanics – minor planets, asteroids – planets and satellites: general

1. Introduction

Lagrange (1867–1892) showed that the three-body problem has five relative equilibrium configurations. In the restricted version (i.e. when one of the bodies is massless) it is useful to view the motion of the test particle in a frame co-rotating with the massive bodies; in this case the relative equilibrium configurations correspond to fixed points two of which, named L_4 and L_5, are linearly stable when the mass ratio is small enough (i.e. when $m_1/m_\mathrm{c} < 0.0385$). These are also known as the triangular equilibrium points and correspond to configurations where the test particle is in the exact 1:1 mean motion resonance, i.e. it is co-orbital[1] with m_1 but leading or trailing it by 60°. In this paper, Trojan-type motion refers to the tadpole-shaped librations around L_4 or L_5 (in analogy with the Trojan asteroids which occupy tadpole orbits in the Sun-Jupiter system) and also to the horseshoe-shaped librations that enclose both triangular points as well as the unstable co-linear point L_3.

In Morais (1999), hereafter Paper I, we constructed a secular theory for Trojan-type motion in the framework of the restricted three-body problem. A secular solution was derived based on the heuristic assumption that the terms in the averaged disturbing potential depending on the relative mean longitude do not have a net effect on the evolution of the eccentricity or inclination. Here we show how we can re-derive this secular solution applying first-order canonical adiabatic theory (Lichtenberg & Lieberman 1983). Moreover, we show that this methodology allows us to construct a complete secular theory which includes the effect of an oblate central mass and the secular perturbations from additional massive bodies, and we are thus able to explain rigorously some preliminary results obtained in Paper I.

2. Settings of the perturbation scheme and the short-term behaviour

2.1. Restricted three-body case

Consider the hierarchical restricted three-body problem with $m_\mathrm{c} \gg m_1$. The equation that describes the motion in the vicinity of L_4 or L_5 is

$$\ddot{\boldsymbol{r}} = -\mathcal{G}\frac{(m_\mathrm{c}+m_1)}{r^3}\boldsymbol{r} + \nabla\mathcal{R} \tag{1}$$

where the disturbing potential $\mathcal{R}$ is defined as

$$\mathcal{R} = \mathcal{G}m_1\left(\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}|} - \frac{\boldsymbol{r}\cdot\boldsymbol{r}_1}{r_1^3} - \frac{1}{r}\right) \tag{2}$$

with $\boldsymbol{r}$ and $\boldsymbol{r}_1$ being, respectively, the vector positions of the test particle and mass m_1 relative to the mass m_c.

Note that the second term on the right hand side of Eq. (1) vanishes at the triangular points; this means that a test particle located at L_4 (or L_5) describes a keplerian ellipse which has an equal size, equal shape and common focus to that described by the mass m_1, but whose orientation is shifted by 60° (or −60°) with respect to the latter.

* e-mail: morais@obs-nice.fr

[1] The word co-orbital, which has here its literal meaning of *shares the orbit with*, is also generally used whenever referring to any nearby orbits.

As usual we define the set (a, e, I, λ, ϖ, Ω) as the osculating elements (respectively, semi-major axis, eccentricity, inclination, mean longitude, longitude of pericentre, longitude of node) of the test particle's orbit. Also, in our notation, unsubscribed quantities refer to the test particle and quantities with subscript k refer to a mass m_k. We then define the average angular velocity around the two-body ellipse (or mean motion) as $n = \dot{\lambda}$, and we relate this to the semi-major axis a by[2]

$$n^2 a^3 = \mathcal{G}(m_\mathrm{c} + m_1). \tag{3}$$

In Paper I we first expressed the disturbing potential $\mathcal{R}$ in terms of osculating elements including terms up to degree two in eccentricities and inclinations; then we performed a change of variables $(\lambda, \lambda_1) \rightarrow (\phi = \lambda - \lambda_1, \lambda_1)$ and averaged $\mathcal{R}$ with respect to the fast angle λ_1. The averaged expansion of the disturbing potential is

$$\begin{aligned}\mathcal{R}_{1:1} \; = \; & n^2 a^2 \mu_1 \frac{a}{a_1}\left(\left[\frac{a^2}{a_1^2} + 1 - 2\frac{a}{a_1}\cos\phi\right]^{-1/2}\right. \\ & \left. - \frac{a}{a_1}\cos\phi - \frac{a_1}{a} + X\right)\end{aligned} \tag{4}$$

with

$$\begin{aligned}X \; = \; & g_1(\phi)e^2 + g_2(\phi)ee_1\cos(\varpi - \varpi_1) \\ & + g_3(\phi)ee_1\sin(\varpi - \varpi_1) \\ & + g_4(\phi)I^2 + g_5(\phi)II_1\cos(\Omega - \Omega_1) \\ & + g_6(\phi)II_1\sin(\Omega - \Omega_1)\end{aligned} \tag{5}$$

where $\mu_1 = m_1/(m_\mathrm{c} + m_1)$, and we refer to Paper I for the explicit form of the functions $g_i(\phi)$.

Tadpole and horseshoe orbits can be decomposed into a slow guiding centre motion described by the variables $\delta a = a - a_1$ and ϕ, with a superimposed fast epicycle due to motion from pericentre to apocentre viewed in the frame co-rotating with m_1. Averaging over the mean longitude λ_1 effectively removes the epicyclic motion and is justified in terms of perturbation theory by the fact that the amplitude of the epicycle is an adiabatic invariant. This is not only confirmed by numerical experiments (e.g. Dermott & Murray 1981) but has also been proved rigorously in the framework of Hill's problem (Henon & Petit 1986). Note, however, that the adiabatic invariance breaks down if the test particle can get close to the mass m_1. Indeed, numerical experiments (Dermott & Murray 1981; Gladman 1993) show that near-circular and near-planar orbits with $0.74\epsilon < \delta a/a_1 < 3.5\epsilon$ (where $\epsilon = (\mu_1/3)^{1/3}$) are strongly affected by successive encounters with m_1, thus being chaotic and potentially unstable. The adiabatic invariance also breaks down in the vicinity of the separatrix between the tadpole and horseshoe regions where there is overlap of high order resonances between the angles ϕ and λ.

[2] Note that Eq. (3) implies that at L_4 (or L_5), $a = a_1$ and $\lambda - \lambda_1 = +60^\circ$ (or -60°) and follows from the fact that the three-body potential at L_4 (or L_5) is of keplerian type but with a larger mean motion.

2.1.1. The guiding centre's motion

We now want to obtain the equation of the guiding centre (that gives the evolution of δa with ϕ). From Lagrange's planetary equations we have

$$\begin{aligned}\dot{\phi} \; = \; & n - n_1 - \frac{2}{na}\frac{\partial \mathcal{R}_{1:1}}{\partial a} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}_{1:1}}{\partial I} \\ & + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2 e}\frac{\partial \mathcal{R}_{1:1}}{\partial e}\end{aligned} \tag{6}$$

$$\delta\dot{a} \; = \; \frac{2}{na}\frac{\partial \mathcal{R}_{1:1}}{\partial \phi}\cdot \tag{7}$$

Following Paper I we note that $\delta a = \mathcal{O}(\sqrt{\mu_1}a_1)$; hence

$$\dot{\phi} = -\frac{3}{2}\frac{\delta a}{a_1}n_1 + \mathcal{O}(\mu_1 n_1). \tag{8}$$

Moreover, if we assume that the eccentricities and the inclinations are small, i.e. at most $\mathcal{O}(\mu_1^{1/4})$, then we have

$$\frac{\delta\dot{a}}{a_1} = 2\mu_1 n_1 \frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi} + \mathcal{O}(\mu_1^{3/2} n_1) \tag{9}$$

where

$$f(\phi) = \frac{1 + 4|\sin(\phi/2)|^3}{2|\sin(\phi/2)|}\cdot \tag{10}$$

From Eqs. (8) and (9), neglecting terms $\mathcal{O}(\mu_1^{3/2} n_1^2)$, we obtain

$$\ddot{\phi} = -3\mu_1 n_1^2 \frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}\cdot \tag{11}$$

This equation (which can also be found in the work of Yoder et al. 1983) has an integral of the motion

$$\mu_1 n_1^2 E = -\frac{1}{6}\dot{\phi}^2 - \mu_1 n_1^2 f(\phi) < 0. \tag{12}$$

Now from Eqs. (8) and (12) we obtain the equation of the guiding centre

$$\frac{\delta a}{a_1} = \pm\sqrt{\frac{8}{3}\mu_1\left(-E - f(\phi)\right)} \tag{13}$$

and as $f(\phi)$ has minima at $\phi = \pm\pi/3$, the maximum value of $|\delta a|/a_1$ occurs at $\phi = \pm\pi/3$ and is given by

$$a_0 = \sqrt{\frac{8}{3}\mu_1\left(-E - \frac{3}{2}\right)}. \tag{14}$$

In Fig. 1 we show some guiding centre trajectories described by Eq. (13). Orbits with $3/2 < -E < 5/2$ are tadpole-shaped and orbits with $-E > 5/2$ are horseshoe-shaped. The separatrix curve (i.e. the boundary between the tadpole and horseshoe regions) has $-E = 5/2$.

Alternatively, we can describe the motion of the guiding centre using the integrable Hamiltonian

$$H_0 = -\frac{3}{8}n_1^2 a_1^2\left(\frac{\delta a}{a_1}\right)^2 - \mu_1 n_1^2 a_1^2 f(\phi) = \mu_1 n_1^2 a_1^2 E \tag{15}$$

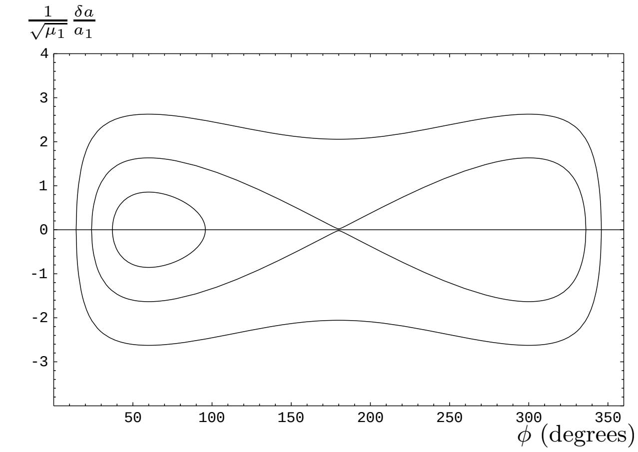

Fig. 1. Guiding centre trajectories obtained with Eq. (13): $-E = 1.77465$ (inner curves), $-E = 2.50019$ (separatrix) and $-E = 4.08606$ (outer curve). Note that $f(\phi)$ is symmetric about $\phi = 180°$; in particular this implies that when $-E = 1.77465$ there is another possible trajectory which is symmetric with respect to the one shown here

whose trajectories have periodicity

$$T_{1:1} = \oint \frac{\mathrm{d}\phi}{\dot{\phi}} \tag{16}$$

and which admits the action-angle variables (J, θ), such that

$$J = H_0 \frac{T_{1:1}}{2\pi} \tag{17}$$

$$\dot{\theta} = \frac{2\pi}{T_{1:1}}. \tag{18}$$

2.1.2. Expansion of the Hamiltonian

The time-independent Hamiltonian of the restricted three-body problem can be written as

$$H = -\frac{(\mathcal{G}(m_\mathrm{c} + m_1))^2}{2\Lambda^2} - \mathcal{R} + \Lambda_1 n_1 \tag{19}$$

where we used the following angle-action variables of the isolated two-body systems

$$\begin{aligned} &\lambda \qquad && \Lambda = \sqrt{\mathcal{G}(m_\mathrm{c} + m_1)a} \\ &\lambda_1 \qquad && \Lambda_1 = \sqrt{\mathcal{G}(m_\mathrm{c} + m_1)a_1}. \end{aligned} \tag{20}$$

In order to obtain Eq. (4) we performed a canonical transformation to the following angle-action variables

$$\begin{aligned} &\hat{\lambda} = \lambda - \lambda_1 \qquad && \hat{\Lambda} = \Lambda \\ &\hat{\lambda}_1 = \lambda_1 \qquad && \hat{\Lambda}_1 = \Lambda_1 + \Lambda. \end{aligned} \tag{21}$$

Therefore, the time-independent Hamiltonian of the averaged restricted three-body problem in the vicinity of the 1:1 mean motion resonance is

$$\hat{H} = -\frac{(\mathcal{G}(m_\mathrm{c} + m_1))^2}{2\hat{\Lambda}^2} - \hat{\Lambda} n_1 + \hat{\Lambda}_1 n_1 - \mathcal{R}_{1:1} \tag{22}$$

and as there is no dependence on $\hat{\lambda}_1$, the third term on the right hand side of Eq. (22) is constant and thus can be dropped.

It is straightforward to obtain again H_0 (given by Eq. (15)) by expanding Eq. (22) in powers of $\delta a/a_1$, assuming that the eccentricities and the inclinations are at most $\mathcal{O}(\mu_1^{1/4})$, and keeping only the terms up to $\mathcal{O}(\mu_1 n_1^2 a_1^2)$ while dropping any constant terms. We can then expand the Hamiltonian of the averaged problem (Eq. (22)) to first-order in the small parameter $\sqrt{\mu_1}$, i.e. we write

$$H = H_0 + \sqrt{\mu_1} H_1 \tag{23}$$

where H_0 is given by Eq. (15) and

$$\sqrt{\mu_1} H_1 = n_1^2 a_1^2 \left(\frac{7}{16} \left(\frac{\delta a}{a_1} \right)^3 + \mu_1 f(\phi) \right) - \mathcal{R}_{1:1}. \tag{24}$$

It is important to recall here the assumption of small eccentricities and inclinations made in the derivation of the guiding centre's equation (Eq. (11)) and in the expansion of the Hamiltonian (Eq. (23)). As under this approximation the zero-order part of the Hamiltonian, H_0, is independent of e or I, we obtained guiding centre trajectories which are stationary with respect to the secular perturbations. Therefore, we do not consider here the co-orbital structures whose guiding centres are modulated by the secular evolution, which have been studied by Namouni (1999) in the framework of the three-body problem.

2.2. The effect of an oblate planet

Now suppose that the body with mass m_c has an oblate shape with equatorial radius R and that we can consider only the dominant zonal harmonic term (with coefficient J_2) in the oblateness disturbing potential. It can be shown (Kozai 1959) that the effect of the oblate planet on a satellite's orbit can be modeled by averaging the disturbing potential over the mean longitude, i.e. by taking into account only the secular part of the oblateness disturbing potential, which is

$$\bar{\mathcal{R}}_\mathrm{obl} = n a^2 \beta(a) \left(\frac{1}{3} - \frac{1}{2} \sin^2 I \right) (1 - e^2)^{-3/2} \tag{25}$$

with

$$\beta(a) = \frac{3}{2} J_2 \left(\frac{R}{a} \right)^2 \sqrt{\mathcal{G} m_\mathrm{c}}\, a^{-3/2}. \tag{26}$$

The potential $\bar{\mathcal{R}}_\mathrm{obl}$ induces the following precession rates in the satellite's orbit

$$\begin{aligned} \dot{\varpi} &= \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial \bar{\mathcal{R}}_\mathrm{obl}}{\partial e} + \frac{\tan \frac{1}{2} I}{na^2 \sqrt{1-e^2}} \frac{\partial \bar{\mathcal{R}}_\mathrm{obl}}{\partial I} \\ &= \beta(a)(1-e^2)^{-2} \left[1 - \frac{3}{2} \sin^2 I - \tan \frac{I}{2} \sin I \cos I \right] \\ &\approx \beta(a) \end{aligned} \tag{27}$$

$$\begin{aligned} \dot{\Omega} &= \frac{1}{na^2 \sqrt{1-e^2} \sin I} \frac{\partial \bar{\mathcal{R}}_\mathrm{obl}}{\partial I} \\ &= -\beta(a)(1-e^2)^{-2} \cos I \approx -\beta(a) \end{aligned} \tag{28}$$

and it also introduces the following correction to the mean motion of the satellite's orbit

$$\dot{\lambda} = n - \frac{2}{na}\frac{\partial \bar{\mathcal{R}}_{\rm obl}}{\partial a} + \frac{\tan\frac{1}{2}I}{na^2\sqrt{1-e^2}}\frac{\partial \bar{\mathcal{R}}_{\rm obl}}{\partial I} + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2 e}\frac{\partial \bar{\mathcal{R}}_{\rm obl}}{\partial e} \approx n + 2\beta(a). \quad (29)$$

We now want to see what the effect of an oblate planet is on the guiding centre's motion of satellites in tadpole or horseshoe orbits. As there are no perturbations on the semi-major axis[3] we only have to take into account the contribution of the secular oblateness potential to $\dot{\phi} = \dot{\lambda} - \dot{\lambda}_1$ which from Eq. (29) is $2\beta(a) - 2\beta(a_1)$. Adding this to the right hand side of Eq. (8) yields

$$\dot{\phi} = -\frac{3}{2}\frac{\delta a}{a_1} n_1 \left(1 + 7J_2\left(\frac{R}{a_1}\right)^2\right) + \mathcal{O}(\mu_1 n_1). \quad (30)$$

Then, from Eqs. (30) and (9), we obtain an integral of the ϕ-motion similar to Eq. (12). Therefore, the motion of the guiding centre is described by an integrable Hamiltonian H_0, which has the same form as Eq. (15). So again we write the Hamiltonian of the averaged problem as

$$H = H_0 + \sqrt{\mu_1} H_1 \quad (31)$$

where now

$$\sqrt{\mu_1} H_1 = n_1^2 a_1^2 \left(\frac{7}{16}\left(\frac{\delta a}{a_1}\right)^3 + \mu_1 f(\phi)\right) - \mathcal{R}_{1:1} - \bar{\mathcal{R}}_{\rm obl}. \quad (32)$$

As the orbit of the mass m_1 precesses due to the oblateness, the Hamiltonian given by Eq. (31) depends explicitly on time. If we assume that $\dot{\varpi} = -\dot{\Omega} = \beta(a) = \beta$ and $\dot{\varpi}_1 = -\dot{\Omega}_1 = \beta(a_1) = \beta_1$, one can avoid this explicit time dependence by introducing the following two extra degrees of freedom and associated conjugate momenta

$$\begin{array}{ll} \lambda_{\rm g1} = -\beta_1 t - \chi_1 & \Lambda_{\rm g1} \\ \lambda_{\rm f1} = \beta_1 t - \Xi_1 & \Lambda_{\rm f1} \end{array} \quad (33)$$

and by adding the terms $-\Lambda_{\rm g1}\beta_1$ and $\Lambda_{\rm f1}\beta_1$ to the new Hamiltonian.

2.3. The effect of additional massive bodies

Now consider N massive bodies orbiting a primary of mass $m_{\rm c}$. It is well known (see e.g. Brouwer & Clemence 1961) that in the absence of mean motion resonances between the N massive bodies, one can obtain a good description of the long-term behaviour of, say m_j, by averaging the disturbing potential over the mean longitudes. The expansion of this secular disturbing potential, up to degree two in eccentricities and inclinations, is

$$\bar{\mathcal{R}}_j = n_j a_j^2 \Big(\frac{A_{j,j}}{2}e_j^2 + \sum_{i\neq j} A_{j,i} e_j e_i \cos(\varpi_j - \varpi_i) + \frac{B_{j,j}}{2} I_j^2 + \sum_{i\neq j} B_{j,i} I_j I_i \cos(\Omega_j - \Omega_i)\Big) \quad (34)$$

where the elements of the matrices A and B are functions of the masses, mean motions and semi-major axes of the massive bodies, whose definition can be found in Paper I.

The secular solution for the mass m_j is simply

$$e_j \exp[{\rm i}\varpi_j] = \sum_i e_{j,i} \exp[{\rm i}(g_i t + \chi_i)] \quad (35)$$

$$I_j \exp[{\rm i}\Omega_j] = \sum_i I_{j,i} \exp[{\rm i}(f_i t + \Xi_i)] \quad (36)$$

where the frequencies g_i (and f_i) are the eigen-values of the matrix A (and B), the coefficients $e_{j,i}$ (and $I_{j,i}$) are the components of the eigen-vectors associated with g_i (and f_i), and the phases χ_i (and Ξ_i) are constants determined by the initial conditions.

Moreover, the secular disturbing potential acting on a test particle in a tadpole or horseshoe orbit associated with the mass m_k, due to the masses m_j (where $j \neq k$), expanded up to degree two in eccentricities and inclinations, is

$$\bar{\mathcal{R}}_{0,k} = na^2 \Big(\frac{A_k}{2}e^2 + \sum_{j\neq k} A_j e e_j \cos(\varpi - \varpi_j) + \frac{B_k}{2} I^2 + \sum_{j\neq k} B_j I I_j \cos(\Omega - \Omega_j)\Big) \quad (37)$$

with A_i (and B_i) defined as $A_{k,i}$ (and $B_{k,i}$) but with $m_k = 0$, a_k replaced by a, and n_k replaced by n.

We now want to see how the secular perturbations from these additional massive bodies change the guiding centre's motion of tadpole or horseshoe orbits. Obviously there are no secular perturbations to the semi-major axis. Now, adding to the right hand side of Eq. (8) the contribution of the additional massive bodies to $\dot{\phi} = \dot{\lambda} - \dot{\lambda}_k$, yields

$$\dot{\phi} = -\frac{3}{2}\frac{\delta a}{a_k} n_k \left(1 + b\sum_{i<k}\frac{m_i}{m_{\rm c}} + \tilde{b}\sum_{i>k}\frac{m_i}{m_{\rm c}}\right) + \mathcal{O}(\mu_k n_k) \quad (38)$$

where b and $\tilde{b}$ are functions of the semi-major axes of the massive bodies.

Then, from Eqs. (38) and (9), we obtain an integral of the ϕ-motion similar to Eq. (12). Therefore, the motion of the guiding centre is still described by an integrable Hamiltonian H_0, which has the same form as Eq. (15). So once more we write the Hamiltonian of the averaged problem as

$$H = H_0 + \sqrt{\mu_k} H_1 \quad (39)$$

[3] Here we refer to a semi-major axis obeying the approximate relation $n^2 a^3 = \mathcal{G} m_{\rm c}(1 + (3/2)J_2(R/a_1)^2)$ which follows from the fact that the oblateness potential is of nearly keplerian type but with a larger mean motion (see Greenberg 1981).

where now

$$\sqrt{\mu_k}H_1 = n_k^2 a_k^2 \left(\frac{7}{16} \left(\frac{\delta a}{a_k} \right)^3 + \mu_k f(\phi) \right) - \mathcal{R}_{1:1} - \bar{\mathcal{R}}_{0,k}. \quad (40)$$

Note that although this Hamiltonian depends explicitly on time through the relations in Eqs. (35) and (36), this can be avoided by over-extending the phase space to include the following extra $2N$ degrees of freedom and associated conjugate momenta

$$\begin{array}{ll} \lambda_{\mathrm{g}i} = -g_i t - \chi_i & \Lambda_{\mathrm{g}i} \\ \lambda_{\mathrm{f}i} = -f_i t - \Xi_i & \Lambda_{\mathrm{f}i} \end{array} \quad (41)$$

and by adding the terms $-\sum_i \Lambda_{\mathrm{g}i} g_i$ and $-\sum_i \Lambda_{\mathrm{f}i} f_i$ to the new Hamiltonian.

3. The long-term behaviour: Secular theory

3.1. Restricted three-body case

We now recall Sect. 2.1.2 where we obtained the Hamiltonian of the averaged problem (Eq. (23)) and then apply Eq. (13) in order to write its first-order part (Eq. (24)) as

$$\sqrt{\mu_1}H_1 \; = \; \mu_1 n_1^2 a_1^2 F(\phi) \frac{\delta a}{a_1} - X \quad (42)$$

where $F(\phi)$ is a function of ϕ,

$$\begin{aligned} X \; = \; & \mu_1 n_1 [g_1(\phi) 2M + g_4(\phi) 2N \\ & + g_2(\phi)\sqrt{n}a(2M)^{1/2} e_1 \cos(\varpi - \varpi_1) \\ & + g_3(\phi)\sqrt{n}a(2M)^{1/2} e_1 \sin(\varpi - \varpi_1) \\ & + g_5(\phi)\sqrt{n}a(2N)^{1/2} I_1 \cos(\Omega - \Omega_1) \\ & + g_6(\phi)\sqrt{n}a(2N)^{1/2} I_1 \sin(\Omega - \Omega_1)] \end{aligned} \quad (43)$$

and

$$\begin{array}{ll} -\varpi & M = \frac{1}{2} n a^2 e^2 \\ -\Omega & N = \frac{1}{2} n a^2 I^2 \end{array} \quad (44)$$

is an approximate canonical set of coordinates and conjugate momenta for the secular Hamiltonian (i.e. the Hamiltonian obtained after the elimination of the ϕ-dependence).

3.1.1. Adiabatic approximation

By construction of the Hamiltonian of the averaged problem (Eq. (23)) the variables $(-\varpi, M)$ and $(-\Omega, N)$ vary on a timescale $\mathcal{O}(\mu_1^{-1/2} T_{1:1})$ and therefore are much slower than θ, the angle variable of the integrable part H_0. The general procedure for dealing with this type of quasi-integrable systems involves a canonical transformation of variables such that the new Hamiltonian does not depend on the fast angle and thus the associated action is an adiabatic invariant (Lichtenberg & Lieberman 1983). To first-order in the small parameter $\sqrt{\mu_1}$, the transformed Hamiltonian is

$$\bar{H} = H_0 + \sqrt{\mu_1} < H_1 >_\theta \quad (45)$$

where

$$< H_1 >_\theta = \frac{1}{2\pi} \int_0^{2\pi} H_1 \, \mathrm{d}\theta = \frac{1}{T_{1:1}} \oint H_1 \frac{\mathrm{d}\phi}{\dot{\phi}}. \quad (46)$$

Now from Eq. (8)

$$\frac{1}{T_{1:1}} \oint \frac{\delta a}{a_1} \frac{\mathrm{d}\phi}{\dot{\phi}} = \mathcal{O}(\mu_1) \quad (47)$$

hence we obtain the transformed Hamiltonian

$$\bar{H} \; = \; H_0 - \bar{X} \quad (48)$$

with

$$\begin{aligned} \bar{X} \; = \; & \mu_1 n_1 [\bar{g}_1[l](h^2 + k^2) + \bar{g}_4[l](p^2 + q^2) \\ & + \bar{g}_2[l]\sqrt{n_1} a_1 e_1 (h \sin\varpi_1 + k \cos\varpi_1) \\ & + \bar{g}_3[l]\sqrt{n_1} a_1 e_1 (h \cos\varpi_1 - k \sin\varpi_1) \\ & + \bar{g}_5[l]\sqrt{n_1} a_1 I_1 (p \sin\Omega_1 + q \cos\Omega_1) \\ & + \bar{g}_6[l]\sqrt{n_1} a_1 I_1 (p \cos\Omega_1 - q \sin\Omega_1)] \end{aligned} \quad (49)$$

where (k, h) and (q, p) are defined as

$$\begin{array}{ll} k = \sqrt{n_1} a_1 e \cos\varpi & h = \sqrt{n_1} a_1 e \sin\varpi \\ q = \sqrt{n_1} a_1 I \cos\Omega & p = \sqrt{n_1} a_1 I \sin\Omega. \end{array} \quad (50)$$

Moreover,

$$g_i[l] = \frac{1}{T_{1:1}} \oint g_i(\phi) \frac{\mathrm{d}\,\phi}{\dot{\phi}} \quad (51)$$

where l is a parameter that characterises the size of the tadpole or horseshoe orbit which we define as

$$l = \pi/3 - \phi_{\min} \quad (52)$$

and $\phi_{\min}$ is the minimal distance to m_1 obtained by solving Eq. (12) with $\dot{\phi} = 0$.

From Paper I we recall that $\bar{g}_5[l] = -2\bar{g}_4[l]$ and $\bar{g}_6[l] = 0$; then we write[4]

$$2\bar{g}_4[l]\mu_1 n_1 \; = \; \Gamma[l] \quad (53)$$

$$2\bar{g}_1[l]\mu_1 n_1 \; = \; \gamma[l] \quad (54)$$

$$\frac{\bar{g}_2[l]}{2\bar{g}_1[l]} \; = \; -c[l] \cos b[l] \quad (55)$$

$$\frac{\bar{g}_3[l]}{2\bar{g}_1[l]} \; = \; -c[l] \sin b[l]. \quad (56)$$

We also note that from Eq. (14) we have

$$a_0[l] = \sqrt{\frac{8}{3}\mu_1 \left(f(\pi/3 - l) - \frac{3}{2} \right)} \quad (57)$$

and we refer to Paper I for plots of $\Gamma[l]$, $\gamma[l]$, $c[l]$ and $b[l]$ as functions of $a_0[l]$.

[4] There is an obvious error in the definition of $c[l]$ in Paper I (Eq. (36)). This only takes the unit value when $l = 0$ and when $l \to \infty$, as can easily be confirmed in Fig. 1b from the same paper.

3.1.2. Secular solution

We now write Hamilton's equations for (k,h) as

$$\begin{aligned}\dot{k}+\mathrm{i}\dot{h} &= -\frac{\partial\bar{X}}{\partial h}+\mathrm{i}\frac{\partial\bar{X}}{\partial k}\\ &= \mathrm{i}\gamma[l](k+\mathrm{i}h)-\mathrm{i}\gamma[l]\sqrt{n_1}a_1c[l]e_1\exp[\mathrm{i}(\varpi_1+b[l])]\end{aligned} \tag{58}$$

which imply the solution for (e,ϖ)

$$\begin{aligned}e\exp[\mathrm{i}\varpi] &= e_{\mathrm{p}}\exp[\mathrm{i}(\gamma[l]t+\chi)]\\ &\quad +c[l]e_1\exp[\mathrm{i}(\varpi_1+b[l])]\end{aligned} \tag{59}$$

i.e. composed of a proper term (proper eccentricity e_{p} and proper precession frequency $\gamma[l]$) and a forced term (forced eccentricity $c[l]e_1$ and forced periapse $b[l]$).

Similarly, we write Hamilton's equations for (q,p) as

$$\begin{aligned}\dot{q}+\mathrm{i}\dot{p} &= -\frac{\partial\bar{X}}{\partial p}+\mathrm{i}\frac{\partial\bar{X}}{\partial q}\\ &= \mathrm{i}\Gamma[l](q+\mathrm{i}p)-\mathrm{i}\Gamma[l]\sqrt{n_1}a_1I_1\exp[\mathrm{i}\Omega_1]\end{aligned} \tag{60}$$

which imply the solution for (I,Ω)

$$I\exp[\mathrm{i}\Omega]=I_{\mathrm{p}}\exp[\mathrm{i}(\Gamma[l]t+\Xi)]+I_1\exp[\mathrm{i}\Omega_1] \tag{61}$$

i.e. composed of a proper term (proper inclination I_{p} and proper precession frequency $\Gamma[l]$) and a forced term I_1 (note, however, that if we choose the orbital plane of m_{c} and m_1 as reference, then this forced term disappears).

3.2. The effect of an oblate planet

The Hamiltonian of the averaged problem has now six degrees of freedom (cf. Sect. 2.2) i.e. two more than in the restricted three-body case. However, we can eliminate these extra degrees of freedom by making a canonical transformation to the following variables

$$\begin{array}{ll}-\varpi_{\mathrm{r}}=-\varpi-\lambda_{\mathrm{g1}} & M\\ -\Omega_{\mathrm{r}}=-\Omega-\lambda_{\mathrm{f1}} & N\\ \lambda_{\mathrm{g1}} & \tilde{\Lambda}_{\mathrm{g1}}=M+\Lambda_{\mathrm{g1}}\\ \lambda_{\mathrm{f1}} & \tilde{\Lambda}_{\mathrm{f1}}=N+\Lambda_{\mathrm{f1}}.\end{array} \tag{62}$$

The transformed first-order part of the Hamiltonian (Eq. (32)) is then

$$\begin{aligned}\sqrt{\mu_1}\tilde{H}_1 &= -\beta_1(\tilde{\Lambda}_{\mathrm{g1}}-\tilde{\Lambda}_{\mathrm{f1}})+\mu_1n_1^2a_1^2F(\phi)\frac{\delta a}{a_1}\\ &\quad -X-Y\end{aligned} \tag{63}$$

with

$$\begin{aligned}X &= \mu_1n[g_1(\phi)2M+g_2(\phi)\sqrt{n}a(2M)^{1/2}e_1\cos\varpi_{\mathrm{r}}\\ &\quad +g_3(\phi)\sqrt{n}a(2M)^{1/2}e_1\sin\varpi_{\mathrm{r}}\\ &\quad +g_4(\phi)2N+g_5(\phi)\sqrt{n}a(2N)^{1/2}I_1\cos\Omega_{\mathrm{r}}\\ &\quad +g_6(\phi)\sqrt{n}a(2N)^{1/2}I_1\sin\Omega_{\mathrm{r}}]\end{aligned} \tag{64}$$

and

$$Y=\beta_1(M-N)\left(1+\frac{\delta a}{a_1}\right)^{-7/2}-\beta_1(M-N). \tag{65}$$

3.2.1. Adiabatic approximation

The first term on the right hand side of Eq. (63) can be dropped as there is no explicit dependence on λ_{g1} or λ_{f1}. Hence, Eq. (63) has the same form as Eq. (42), except for the term Y. This introduces the relative precession frequencies

$$\frac{\partial Y}{\partial M}=-\frac{\partial Y}{\partial N}\approx-\frac{7}{2}\beta_1\frac{\delta a}{a_1}=\mathcal{O}(\beta_1\sqrt{\mu_1}). \tag{66}$$

As in general $\beta_1\ll n_1$, the variables $(-\varpi_{\mathrm{r}},M)$ and $(-\Omega_{\mathrm{r}},N)$ still vary on a time-scale much longer than the co-orbital period $T_{1:1}=\mathcal{O}(n_1\sqrt{\mu_1})$. We can then follow the procedure described previously, obtaining an Hamiltonian as defined by Eqs. (45) and (46), i.e.

$$\bar{H}=H_0-\bar{X}-\bar{Y} \tag{67}$$

with

$$\begin{aligned}\bar{X} &= \mu_1n_1[\bar{g}_1[l](h_{\mathrm{r}}^2+k_{\mathrm{r}}^2)+\bar{g}_4[l](p_{\mathrm{r}}^2+q_{\mathrm{r}}^2)\\ &\quad +\bar{g}_2[l]\sqrt{n_1}a_1e_1k_{\mathrm{r}}+\bar{g}_3[l]\sqrt{n_1}a_1e_1h_{\mathrm{r}}\\ &\quad +\bar{g}_5[l]\sqrt{n_1}a_1I_1q_{\mathrm{r}}+\bar{g}_6[l]\sqrt{n_1}a_1I_1p_{\mathrm{r}}]\end{aligned} \tag{68}$$

and

$$\bar{Y}=\frac{1}{2}\Delta\bar{\beta}[(h_{\mathrm{r}}^2+k_{\mathrm{r}}^2)-(p_{\mathrm{r}}^2+q_{\mathrm{r}}^2)] \tag{69}$$

where $(k_{\mathrm{r}},h_{\mathrm{r}})$ and $(q_{\mathrm{r}},p_{\mathrm{r}})$ are defined as

$$\begin{array}{ll}k_{\mathrm{r}}=\sqrt{n_1}a_1e\cos\varpi_{\mathrm{r}} & h_{\mathrm{r}}=\sqrt{n_1}a_1e\sin\varpi_{\mathrm{r}}\\ q_{\mathrm{r}}=\sqrt{n_1}a_1I\cos\Omega_{\mathrm{r}} & p_{\mathrm{r}}=\sqrt{n_1}a_1I\sin\Omega_{\mathrm{r}}\end{array} \tag{70}$$

and from Eq. (30)

$$\Delta\bar{\beta}=-\beta_1+\frac{\beta_1}{T_{1:1}}\oint\left(1+\frac{\delta a}{a_1}\right)^{-7/2}\frac{\mathrm{d}\phi}{\dot{\phi}}=\mathcal{O}(\beta_1\mu_1). \tag{71}$$

3.2.2. Secular solution

Hamilton's equations for $(k_{\mathrm{r}},h_{\mathrm{r}})$ are

$$\begin{aligned}&\dot{k}_{\mathrm{r}}+\mathrm{i}\dot{h}_{\mathrm{r}}=\left(-\frac{\partial}{\partial h_{\mathrm{r}}}+\mathrm{i}\frac{\partial}{\partial k_{\mathrm{r}}}\right)(\bar{X}+\bar{Y})=\\ &\mathrm{i}(\gamma[l]+\Delta\bar{\beta})(k_{\mathrm{r}}+\mathrm{i}h_{\mathrm{r}})-\mathrm{i}\gamma[l]\sqrt{n_1}a_1c[l]e_1\exp[\mathrm{i}b[l]]\end{aligned} \tag{72}$$

and therefore the solution for (e,ϖ_{r}) is

$$\begin{aligned}e\exp[\mathrm{i}\varpi_{\mathrm{r}}] &= e_{\mathrm{p}}\exp[\mathrm{i}((\gamma[l]+\Delta\bar{\beta})t+\chi)]\\ &\quad +\frac{\gamma[l]c[l]e_1\exp[\mathrm{i}b[l]]}{\gamma[l]+\Delta\bar{\beta}}.\end{aligned} \tag{73}$$

Similarly, Hamilton's equations for $(q_{\mathrm{r}},p_{\mathrm{r}})$ are

$$\begin{aligned}\dot{q}_{\mathrm{r}}+\mathrm{i}\dot{p}_{\mathrm{r}} &= \left(-\frac{\partial}{\partial p_{\mathrm{r}}}+\mathrm{i}\frac{\partial}{\partial q_{\mathrm{r}}}\right)(\bar{X}+\bar{Y})\\ &= \mathrm{i}(\Gamma[l]-\Delta\bar{\beta})(q_{\mathrm{r}}+\mathrm{i}p_{\mathrm{r}})-\mathrm{i}\Gamma[l]\sqrt{n_1}a_1I_1\end{aligned} \tag{74}$$

and therefore the solution for (I,Ω_{r}) is

$$\begin{aligned}I\exp[\mathrm{i}\Omega_{\mathrm{r}}] &= I_{\mathrm{p}}\exp[\mathrm{i}((\Gamma[l]-\Delta\bar{\beta})t+\Xi)]\\ &\quad +\frac{\Gamma[l]I_1}{\Gamma[l]-\Delta\bar{\beta}}.\end{aligned} \tag{75}$$

From Eqs. (73) and (75) we see that the oblateness term, $\Delta\bar{\beta} > 0$, leads to a decrease in the secular precession periods and the magnitude of the forced terms. However, this effect is only significant if $\Delta\bar{\beta} \sim \gamma$ (or $-\Gamma$). As $\gamma \geq 3.375 n_1 \mu_1$ (see Fig. 1a in Paper I) and in general $\beta_1 \ll n_1$, one can largely neglect the term $\Delta\bar{\beta}$ in that which concerns the evolution of the eccentricity; in this case the secular solution reduces to Eq. (59). On the other hand, very small amplitude tadpole orbits can have $\Gamma \sim -\beta_1\mu_1$ (see Fig. 1d in Paper I) and thus the effect of the term $\Delta\bar{\beta}$ on the evolution of the inclination can be visible.

3.3. The effect of additional massive bodies

The Hamiltonian of the averaged problem has $2 \times N + 4$ degrees of freedom (cf. Sect. 2.3). In order to reduce the number of degrees of freedom we perform a canonical transformation to the following variables

$$\begin{array}{ll} -\varpi_{\rm r} = -\varpi - \lambda_{{\rm g}k} & M \\ -\Omega_{\rm r} = -\Omega - \lambda_{{\rm f}k} & N \\ \lambda_{{\rm g}k} & \tilde{\Lambda}_{{\rm g}k} = M + \Lambda_{{\rm g}k} \\ \lambda_{{\rm f}k} & \tilde{\Lambda}_{{\rm f}k} = N + \Lambda_{{\rm f}k}. \end{array} \quad (76)$$

We also perform $N-1$ canonical transformations of the same type to the following variables

$$\begin{array}{ll} \check{\lambda}_{{\rm g}i} = \lambda_{{\rm g}i} - \lambda_{{\rm g}k} & \Lambda_{{\rm g}i} \\ \check{\lambda}_{{\rm f}i} = \lambda_{{\rm f}i} - \lambda_{{\rm f}k} & \Lambda_{{\rm f}i} \\ \lambda_{{\rm g}k} & \check{\Lambda}_{{\rm g}k} = \sum_{i\neq k}\Lambda_{{\rm g}i} + \tilde{\Lambda}_{{\rm g}k} \\ \lambda_{{\rm f}k} & \check{\Lambda}_{{\rm f}k} = \sum_{i\neq k}\Lambda_{{\rm f}i} + \tilde{\Lambda}_{{\rm f}k}. \end{array} \quad (77)$$

Then, the transformed first-order part of the Hamiltonian (Eq. (40)) is

$$\begin{aligned} \sqrt{\mu_k}\check{H}_1 &= -\check{\Lambda}_{{\rm g}k} g_k - \check{\Lambda}_{{\rm f}k} f_k \\ &\quad - \sum_{i\neq k}\Lambda_{{\rm g}i}(g_i - g_k) - \sum_{i\neq k}\Lambda_{{\rm f}i}(f_i - f_k) \\ &\quad + \mu_k n_k^2 a_k^2 F(\phi)\frac{\delta a}{a_k} - X - Y \end{aligned} \quad (78)$$

with

$$\begin{aligned} X &= \mu_k n[g_1(\phi) 2M + g_4(\phi) 2N \\ &\quad + g_2(\phi)\sqrt{n}a(2M)^{1/2}\sum_i e_{k,i}\cos(\varpi_{\rm r} + \check{\lambda}_{{\rm g}i}) \\ &\quad + g_3(\phi)\sqrt{n}a(2M)^{1/2}\sum_i e_{k,i}\sin(\varpi_{\rm r} + \check{\lambda}_{{\rm g}i}) \\ &\quad + g_5(\phi)\sqrt{n}a(2N)^{1/2}\sum_i I_{k,i}\cos(\Omega_{\rm r} + \check{\lambda}_{{\rm f}i}) \\ &\quad + g_6(\phi)\sqrt{n}a(2N)^{1/2}\sum_i I_{k,i}\sin(\Omega_{\rm r} + \check{\lambda}_{{\rm f}i})] \end{aligned} \quad (79)$$

and

$$\begin{aligned} Y &= (A_k - g_k)M + (B_k - f_k)N \\ &\quad + \sum_{j\neq k} A_j\sqrt{n}a(2M)^{1/2}\sum_i e_{j,i}\cos(\varpi_{\rm r} + \check{\lambda}_{{\rm g}i}) \\ &\quad + \sum_{j\neq k} B_j\sqrt{n}a(2N)^{1/2}\sum_i I_{j,i}\cos(\Omega_{\rm r} + \check{\lambda}_{{\rm f}i}). \end{aligned} \quad (80)$$

And the first two terms on the right hand side of Eq. (78) can be dropped as there is no explicit dependence on $\lambda_{{\rm g}k}$ or $\lambda_{{\rm f}k}$.

3.3.1. Secular resonances

We now assume that the perturbers move on circular and co-planar orbits, in which case the terms in Eq. (78) depending on $(\check{\lambda}_{{\rm g}i}, \Lambda_{{\rm g}i})$ and $(\check{\lambda}_{{\rm f}i}, \Lambda_{{\rm f}i})$ also disappear and the Hamiltonian reduces to

$$\begin{aligned} H &= H_0 + \mu_k n_k^2 a_k^2 F(\phi)\frac{\delta a}{a_k} \\ &\quad - (2g_1(\phi)\mu_k n + A_k - g_k)M \\ &\quad - (2g_4(\phi)\mu_k n + B_k - f_k)N. \end{aligned} \quad (81)$$

Equation (81) represents a one-degree of freedom (hence integrable) system depending on the parameters M and N. Therefore, we can perform a canonical transformation to action-angle variables which eliminates ϕ from the Hamiltonian (Lichtenberg & Lieberman 1983). To first-order in the small parameter $\sqrt{\mu_k}$, this is accomplished by averaging over the angle variable of the zero-order term H_0, which is equivalent to averaging over the co-orbital period $T_{1:1}$. The transformed Hamiltonian is

$$\bar{H} = H_0 - (\gamma_k[l] + \bar{A}_k - g_k)M - (\Gamma_k[l] + \bar{B}_k - f_k)N \quad (82)$$

which has proper frequencies

$$\dot{\varpi}_{\rm r} = -\frac{\partial \bar{H}}{\partial M} = \gamma_k[l] + \bar{A}_k - g_k \quad (83)$$

$$\dot{\Omega}_{\rm r} = -\frac{\partial \bar{H}}{\partial N} = \Gamma_k[l] + \bar{B}_k - f_k \quad (84)$$

where $\gamma_k[l] = 2\bar{g}_1[l]\mu_k n_k$, $\Gamma_k[l] = 2\bar{g}_4[l]\mu_k n_k$; and from Eq. (38)

$$\bar{A}_k = \frac{1}{T_{1:1}}\oint A_k\frac{{\rm d}\phi}{\dot{\phi}} = A_{k,k}(1 + \mathcal{O}(\mu_k)) \quad (85)$$

$$\bar{B}_k = \frac{1}{T_{1:1}}\oint B_k\frac{{\rm d}\phi}{\dot{\phi}} = B_{k,k}(1 + \mathcal{O}(\mu_k)). \quad (86)$$

Secular resonances involving the pericentres (or nodes) occur when $\dot{\varpi}_{\rm r}$ (or $\dot{\Omega}_{\rm r}$) is equal to a forcing frequency $g_i - g_k$ (or $f_i - f_k$), i.e. when

$$\gamma_k[l] + \bar{A}_k = g_i \quad (87)$$

$$\Gamma_k[l] + \bar{B}_k = f_i. \quad (88)$$

Note that the proper frequencies of precession of the Trojan orbit are in fact $\dot{\varpi} = \dot{\varpi}_{\rm r} + g_k$ and $\dot{\Omega} = \dot{\Omega}_{\rm r} + f_k$ which coincide respectively with the left hand side of Eq. (87) and Eq. (88). Moreover, the terms γ_k (and Γ_k) and the terms $\bar{A}_k$ (and $\bar{B}_k$) are respectively the contribution from the mass m_k and the contribution from the additional massive bodies m_j (where $j \neq k$).

3.3.2. Adiabatic approximation

We will now assume that not only the variables $(-\varpi_{\rm r}, M)$ and $(-\Omega_{\rm r}, N)$ but also the angles $\check\lambda_{{\rm g}i}$ and $\check\lambda_{{\rm f}i}$ vary on a time-scale much longer than the co-orbital period $T_{1:1}$. Note that while the first assumption will in general be true, the same does not necessarily apply to the second assumption. When the co-orbital frequency is comparable to one of the forcing frequencies, a low order resonance can occur in which case the adiabatic approximation does not provide a good description of the system. Nevertheless, if we ignore this possibility then we can apply Eqs. (45) and (46) to obtain the transformed Hamiltonian

$$\begin{aligned}\bar H &= H_0 - \sum_{i\neq k}\Lambda_{{\rm g}i}(g_i-g_k) - \sum_{i\neq k}\Lambda_{{\rm f}i}(f_i-f_k)\\ &\quad -\bar X - \bar Y \end{aligned}\tag{89}$$

with

$$\begin{aligned}\bar X &= \mu_k n_k[\bar g_1[l](h_{\rm r}^2+k_{\rm r}^2)+\bar g_4[l](p_{\rm r}^2+q_{\rm r}^2)\\ &\quad +\bar g_2[l]\sqrt{n_k}a_k\sum_i e_{k,i}(k_{\rm r}\cos\check\lambda_{{\rm g}i}-h_{\rm r}\sin\check\lambda_{{\rm g}i})\\ &\quad +\bar g_3[l]\sqrt{n_k}a_k\sum_i e_{k,i}(h_{\rm r}\cos\check\lambda_{{\rm g}i}+k_{\rm r}\sin\check\lambda_{{\rm g}i})\\ &\quad +\bar g_5[l]\sqrt{n_k}a_k\sum_i I_{k,i}(q_{\rm r}\cos\check\lambda_{{\rm f}i}-p_{\rm r}\sin\check\lambda_{{\rm f}i})\\ &\quad +\bar g_6[l]\sqrt{n_k}a_k\sum_i I_{k,i}(p_{\rm r}\cos\check\lambda_{{\rm f}i}+q_{\rm r}\sin\check\lambda_{{\rm f}i})]\end{aligned}\tag{90}$$

and

$$\begin{aligned}\bar Y &= \frac{1}{2}(\bar A_k-g_k)(h_{\rm r}^2+k_{\rm r}^2)+\frac{1}{2}(\bar B_k-f_k)(p_{\rm r}^2+q_{\rm r}^2)\\ &\quad +\sum_{j\neq k}\bar A_j\sqrt{n_j}a_j\sum_i e_{j,i}(k_{\rm r}\cos\check\lambda_{{\rm g}i}-h_{\rm r}\sin\check\lambda_{{\rm g}i})\\ &\quad +\sum_{j\neq k}\bar B_j\sqrt{n_j}a_j\sum_i I_{j,i}(q_{\rm r}\cos\check\lambda_{{\rm f}i}-p_{\rm r}\sin\check\lambda_{{\rm f}i})\end{aligned}\tag{91}$$

where $(k_{\rm r}, h_{\rm r})$ and $(q_{\rm r}, p_{\rm r})$ are defined as

$$k_{\rm r}+{\rm i}h_{\rm r} = \sqrt{n_k}a_k e\exp[{\rm i}\varpi_{\rm r}]\tag{92}$$
$$q_{\rm r}+{\rm i}p_{\rm r} = \sqrt{n_k}a_k I\exp[{\rm i}\Omega_{\rm r}]\tag{93}$$

and from Eq. (38)

$$\bar A_j = \frac{1}{T_{1:1}}\oint A_j\frac{{\rm d}\phi}{\dot\phi} = A_{k,j}(1+\mathcal{O}(\mu_k))\tag{94}$$
$$\bar B_j = \frac{1}{T_{1:1}}\oint B_j\frac{{\rm d}\phi}{\dot\phi} = B_{k,j}(1+\mathcal{O}(\mu_k)).\tag{95}$$

3.3.3. Secular solution

The evolution of $z=\sqrt{n_k}a_k e\exp[{\rm i}\varpi]$ is described by

$$\begin{aligned}&(\dot z-{\rm i}g_k z)\exp[{\rm i}\lambda_{{\rm g}k}] = \left(-\frac{\partial}{\partial h_{\rm r}}+{\rm i}\frac{\partial}{\partial k_{\rm r}}\right)(\bar X+\bar Y) =\\ &{\rm i}\Bigg((\gamma_k[l]+\bar A_k-g_k)z-\gamma_k[l]\sqrt{n_k}a_k c[l]e_k\exp[{\rm i}(\varpi_k+b[l])]\\ &+\sum_{j\neq k}\bar A_j\sqrt{n_j}a_j e_j\exp[{\rm i}\varpi_j]\Bigg)\exp[{\rm i}\lambda_{{\rm g}k}]\end{aligned}\tag{96}$$

which is the equation of a forced harmonic oscillator with proper frequency $\gamma_k[l]+\bar A_k$ and forcing frequencies g_i; hence the solution for (e,ϖ) is

$$\begin{aligned}&e\exp[{\rm i}\varpi] = e_{\rm p}\exp[{\rm i}((\gamma_k[l]+\bar A_k)t+\chi)]\\ &+\sum_i\frac{\gamma_k[l]c[l]e_{k,i}}{\gamma_k[l]+\bar A_k-g_i}\exp[{\rm i}(g_i t+\chi_i+b[l])]\\ &-\sum_i\frac{\sum_{j\neq k}\bar A_j e_{j,i}}{\gamma_k[l]+\bar A_k-g_i}\exp[{\rm i}(g_i t+\chi_i)].\end{aligned}\tag{97}$$

The evolution of $Z=\sqrt{n_k}a_k I\exp[{\rm i}\Omega]$ is described by

$$\begin{aligned}&(\dot Z-{\rm i}f_k Z)\exp[{\rm i}\lambda_{{\rm f}k}] = \left(-\frac{\partial}{\partial p_{\rm r}}+{\rm i}\frac{\partial}{\partial q_{\rm r}}\right)(\bar X+\bar Y)\\ &={\rm i}\Bigg((\Gamma_k[l]+\bar B_k-f_k)Z-\Gamma_k[l]\sqrt{n_k}a_k I_k\exp[{\rm i}\Omega_k]\\ &+\sum_{j\neq k}\bar B_j\sqrt{n_j}a_j I_j\exp[{\rm i}\Omega_j]\Bigg)\exp[{\rm i}\lambda_{{\rm f}k}]\end{aligned}\tag{98}$$

which is the equation of a forced harmonic oscillator with proper frequency $\Gamma_k[l]+\bar B_k$ and forcing frequencies f_i; hence the solution for (I,Ω) is

$$\begin{aligned}&I\exp[{\rm i}\Omega] = I_{\rm p}\exp[{\rm i}((\Gamma_k[l]+\bar B_k)t+\Xi)]\\ &+\sum_i\frac{\Gamma_k[l]I_{k,i}-\sum_{j\neq k}\bar B_j I_{j,i}}{\Gamma_k[l]+\bar B_k-f_i}\exp[{\rm i}(f_i t+\Xi_i)].\end{aligned}\tag{99}$$

We now recall from Paper I that $\bar B_j=(1+\mathcal{O}(\mu_k))B_{k,j}$, and that by definition of eigen-values and eigen-vectors, $B_{k,k}I_{k,i}+\sum_{j\neq k}B_{k,j}I_{j,i}=f_i I_{k,i}$, so that

$$\begin{aligned}I\exp[{\rm i}\Omega] &= I_{\rm p}\exp[{\rm i}((\Gamma_k[l]+\bar B_k)t+\Xi)]\\ &\quad +\sum_i I_{k,i}\exp[{\rm i}(f_i t+\Xi_i)]\\ &\quad +\sum_i\frac{\mathcal{O}(\mu_k\mu_i)I_{k,i}}{\Gamma_k[l]+\bar B_k-f_i}\exp[{\rm i}(f_i t+\Xi_i)].\end{aligned}\tag{100}$$

This is to say that when $\Gamma_k[l]+\bar B_k=f_i$, the forcing terms in the second-order differential equation that describes the evolution of $I\exp[{\rm i}\Omega]$ are smaller than in the non-resonant case by a mass ratio factor. Indeed, one can show that these forcing terms have amplitudes $I_{k,i}(\Gamma_k[l]+\bar B_k+f_i)[(\Gamma_k[l]+\bar B_k-f_i)+\mathcal{O}(\mu_k\mu_i)]$, which reduce to $2f_i I_{k,i}\mathcal{O}(\mu_k\mu_i)$ at the exact resonance.

Note that although the secular solution obtained here (Eqs. (97) and (99)) has essentially the same form as that obtained in Paper I, the terms $\bar A_j$ and $\bar B_j$ are now defined as averages over $T_{1:1}$ (which nonetheless coincide, to lowest order, with the quantities defined in Paper I). The basic improvement with respect to Paper I is the correct derivation of the thresholds of validity of the secular solution which as we have seen now simply depends on the validity of the adiabatic approximation.

Table 1. Uranian satellite system: co-orbital frequencies $\sqrt{(27/4)\mu_i} n_i$ and forcing frequencies $g_i - g_j$ and $f_i - f_j$ (units are rads/day). Note that $g_i - g_j$ and $f_j - f_i$ are nearly equal; this is due to the fact that g_k and f_k are respectively the eigenvalues of matrices A and B which by definition have $A_{kk} = -B_{kk}$ and which are also nearly diagonal (the diagonal term is mostly due to the oblateness and therefore $A_{kk} = -B_{kk} \approx \beta(a_k)$)

i	$\sqrt{(27/4)\mu_i} n_i$	$g_1 - g_i$ $f_1 - f_i$	$g_2 - g_i$ $f_2 - f_i$	$g_3 - g_i$ $f_3 - f_i$	$g_4 - g_i$ $f_4 - f_i$	$g_5 - g_i$ $f_5 - f_i$
1	$1.1\ 10^{-2}$	0	$-7.0\ 10^{-4}$	$-8.5\ 10^{-4}$	$-9.1\ 10^{-4}$	$-9.7\ 10^{-4}$
		0	$7.0\ 10^{-4}$	$8.5\ 10^{-4}$	$9.1\ 10^{-4}$	$9.7\ 10^{-4}$
2	$2.9\ 10^{-2}$	$7.0\ 10^{-4}$	0	$-1.5\ 10^{-4}$	$-2.1\ 10^{-4}$	$-2.7\ 10^{-4}$
		$-7.0\ 10^{-4}$	0	$1.5\ 10^{-4}$	$2.1\ 10^{-4}$	$2.8\ 10^{-4}$
3	$1.3\ 10^{-2}$	$8.5\ 10^{-4}$	$1.5\ 10^{-4}$	0	$-6.0\ 10^{-5}$	$-1.2\ 10^{-4}$
		$-8.5\ 10^{-4}$	$-1.5\ 10^{-4}$	0	$5.4\ 10^{-5}$	$1.2\ 10^{-4}$
4	$1.1\ 10^{-2}$	$9.1\ 10^{-4}$	$2.1\ 10^{-4}$	$6.0\ 10^{-5}$	0	$-6.1\ 10^{-5}$
		$-9.0\ 10^{-4}$	$-2.1\ 10^{-4}$	$-5.4\ 10^{-5}$	0	$6.9\ 10^{-5}$
5	$7.0\ 10^{-3}$	$9.7\ 10^{-4}$	$2.7\ 10^{-4}$	$1.2\ 10^{-4}$	$6.1\ 10^{-5}$	0
		$-9.7\ 10^{-4}$	$-2.8\ 10^{-4}$	$-1.2\ 10^{-4}$	$-6.9\ 10^{-5}$	0

4. Applications

4.1. The uranian satellite system

We will now apply our secular theory to the case of Trojan orbits associated with the five major uranian satellites. As already noticed in Paper I, this system provides a good testing ground for our theory due to two reasons. First: there are no mean motion resonances amongst these satellites and thus the secular approximation (i.e. the average of the disturbing potential over the mean longitudes) is valid. Second: these satellites have nearly-circular and nearly-equatorial orbits thus satisfying the requirements of our secular theory.

We used the data from Malhotra et al. (1989) to calculate the parameters of the secular theory for the uranian system[5] and then identified the modes that satisfy Eq. (87) (or Eq. (88)) inside the co-orbital regions of its major satellites. For instance, in order to locate secular resonances involving the pericentres (or nodes) inside the co-orbital region of the mass m_k, we first plot γ_k (or Γ_k)[6] as a function of a_0 (a parameter that characterizes the size of the tadpole or horseshoe orbit already defined in Eq. (14)) and then identify the intersection with the horizontal lines taken at $g_i - \bar{A}_k$ (or $f_i - \bar{B}_k$). We show the location of the secular resonances which occur inside the co-orbital regions of Miranda (m_1), Ariel (m_2), Umbriel (m_3), Titania (m_4) and Oberon (m_5) in Fig. 2. Note that we can condense the results for the five satellites in one single picture due to the fact that both γ_k and Γ_k scale as $\mu_k n_k$ (see also Paper I).

[5] Here, we took into account Uranus' oblate shape by adding the terms $\beta(a_k)$ (recall definition of function β in Eq. (26)) to A_{kk} and $-B_{kk}$ (the diagonal terms of matrices A and -B which we introduced in Sect. 2.3).

[6] In our calculations we ignored the contribution $\Delta\bar{\beta}$ (given by Eq. (71)) due to Uranus' oblate shape as in general this is unimportant (cf. last paragraph of Sect. 3.2.2).

We also used a Runge-Kutta-Nystron 12th order scheme (Brankin et al. 1987) to integrate the equations of motion of the system consisting of Uranus, its satellites and associated test particles in tadpole or horseshoe orbits. The parameters and initial conditions were again taken from Malhotra et al. (1989), and we incorporated an oblateness potential which takes into account only the dominant zonal harmonic.

In Paper I we showed the result of a numerical integration for a test particle located near the L_4 point of Titania (m_4), which included the gravitational interaction with Umbriel (m_3) only. We saw that in this case the long-term behaviour of (e, ϖ) is affected by the proximity of a secular resonance involving the mode g_3 and is in good agreement with the secular solution (Eq. (97)). In fact, we can see from Table 1 that the adiabatic approximation made in Sect. 3.3.2 should be valid in the case of the uranian satellites, as the co-orbital frequencies are much larger than the forcing frequencies.

In Fig. 3 we show the evolution of an orbit located near the L_4 point of Oberon with $a_0 = 0$. This is very close to the secular resonance involving the mode g_4 (cf. Fig. 2) and we see that the slow periodic motion of the critical argument $\varpi - g_4 t$ is correlated with a large amplitude oscillation of the eccentricity. The small amplitude fast oscillation in the eccentricity has the same periodicity as the argument $\varpi - g_5 t$ and is indeed caused by the forced term due to the 1:1 mean motion resonance with Oberon. There is no apparent threat to the long-term stability of this orbit, as the maximum eccentricity is still very far from the threshold required for close approaches with nearby Titania (i.e. $e = 0.25$).

In Fig. 4 we show the evolution of a horseshoe orbit associated with Oberon which has $a_0 = 2\sqrt{\mu_5}$. The eccentricity exhibits very irregular behaviour which is probably due to the interaction between two nearby secular resonances (involving the modes g_3 and g_4; cf. Fig. 2) as

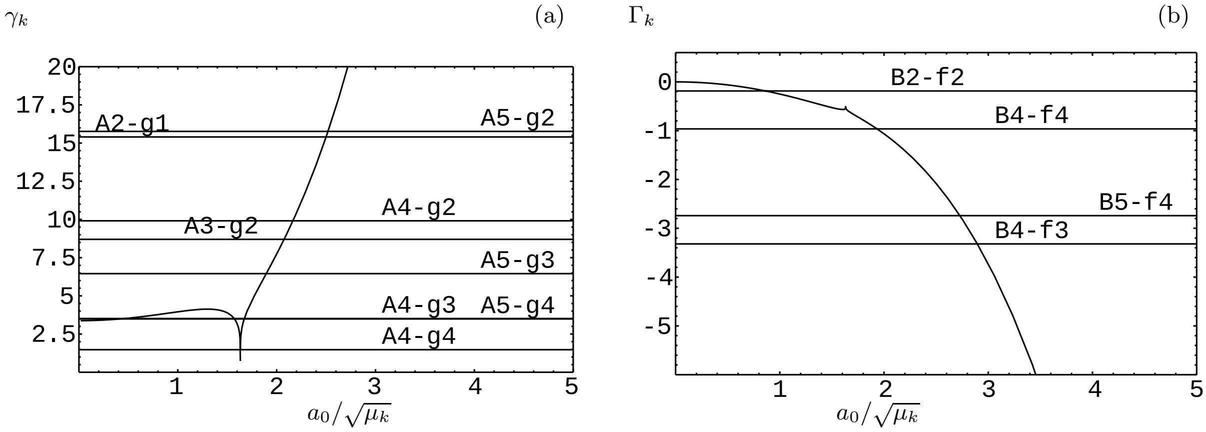

Fig. 2. Location of secular resonances involving **a)** the pericentres and **b)** the nodes, inside the co-orbital regions of the uranian satellites Miranda (m_1), Ariel (m_2), Umbriel (m_3), Titania (m_4) and Oberon (m_5). Secular resonances associated with the modes g_i (or f_i) occur at the locations $a_0/\sqrt{\mu_k}$ determined by the intersections of the curves γ_k (or Γ_k) with the lines labeled $A_k - g_i$ (or $B_k - f_i$). Note that the frequencies γ_k and Γ_k were divided by $\mu_k n_k$ in order to be able to condense the results for all the five satellites in one single picture. The singularity in γ_k and Γ_k at $a_0 = \sqrt{(8/3)\mu_1}$ corresponds to the tadpole-horseshoe separatrix

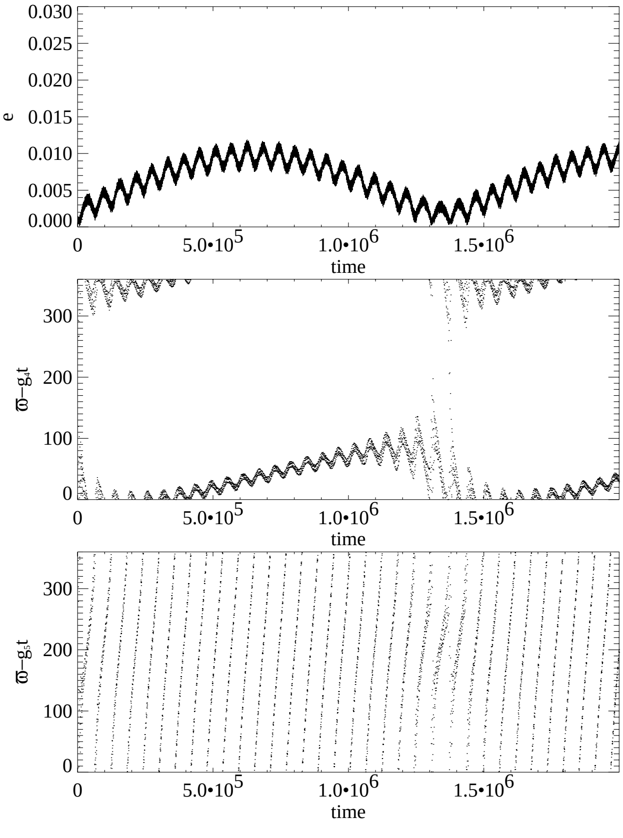

Fig. 3. Evolution of eccentricity (upper figure), critical arguments associated with the modes g_4 (middle figure) and g_5 (lower figure) for an orbit located near Oberon's L_4 point with $a_0 = 0$. Time is in units of Miranda's orbital period

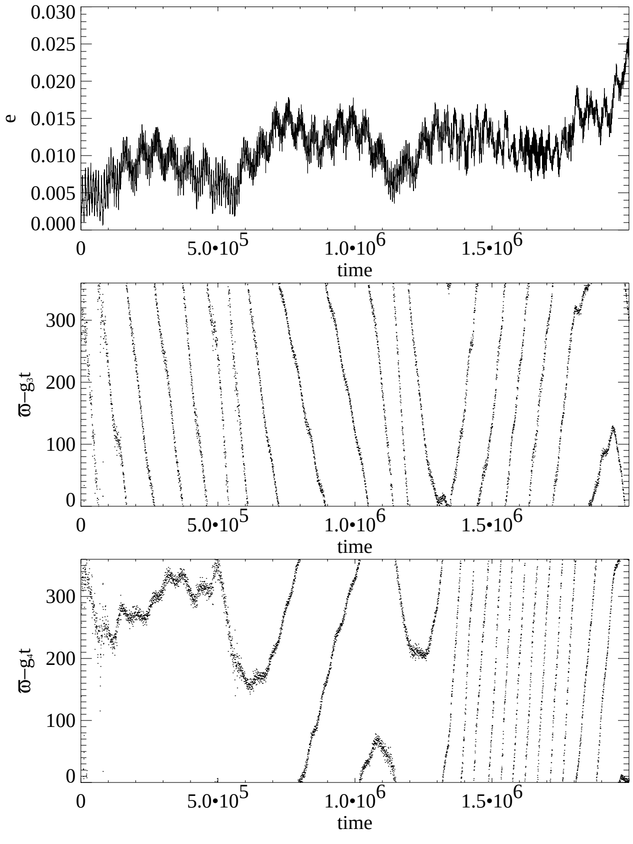

Fig. 4. Evolution of eccentricity (upper figure), critical arguments associated with the modes g_3 (middle figure) and g_4 (lower figure) for a horseshoe orbit associated with Oberon with $a_0 = 2\sqrt{\mu_5}$. Time is in units of Miranda's orbital period

suggested by the behaviour of the associated critical arguments. Due to the overlap of the separatrices of these two secular resonances, the eccentricity diffuses chaotically and can potentially reach stability-threatening values.

We remark here that although our secular theory cannot describe accurately the true behaviour in the very close vicinity of secular resonances (due to the occurrence of singularities in Eqs. (97) and (99) which is an artifact

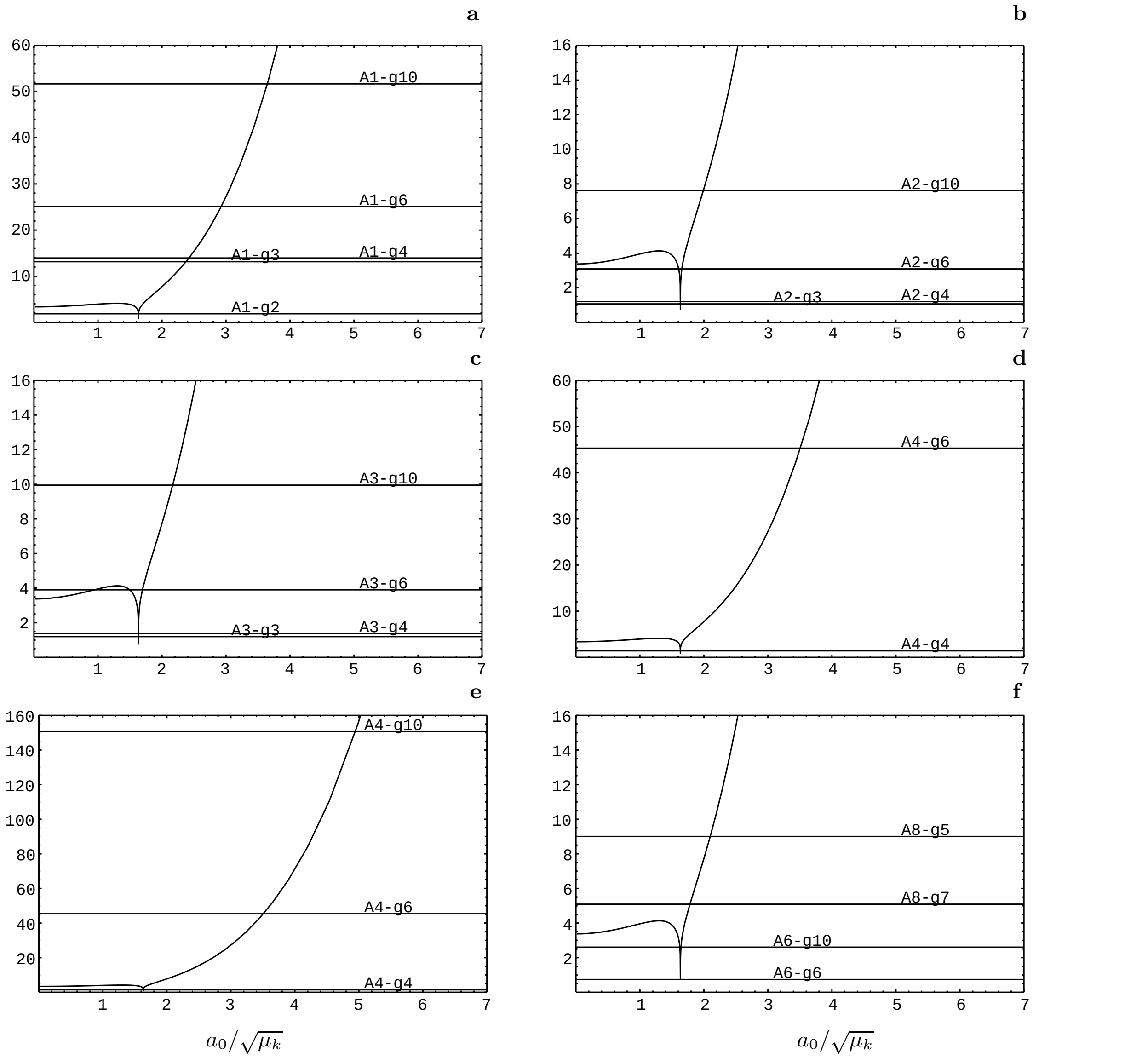

Fig. 5. Location of secular resonances involving the pericentres inside the co-orbital regions of the planets. The frequencies shown in the vertical axes are adimensional and correspond to the true frequencies divided by $\mu_k n_k$. Secular resonances associated with the modes g_i occur at the locations $a_0/\sqrt{\mu_k}$ determined by the intersections of the lines labeled $A_k - g_i$ with the curves γ_k. The secular resonances involving the modes g_1 and g_2 (not shown here as they are likely to be very weak) also occur inside the co-orbital regions of Uranus and Neptune

caused by the truncation of the disturbing potential at degree two in e and I) or whenever there is overlap of adjacent secular resonances (due to the underlying chaotic nature of the phase space), it is still very useful in the sense that it allows us to obtain the location of these secular resonances, whose dynamical effect can always be subsequently investigated with numerical integrations.

4.2. Our planetary system

We also applied our secular theory to the system consisting of the eight planets, Mercury to Neptune, using values for the eigen-frequencies of the secular system from Brouwer & Van Woerkom (1950). As these were calculated taking in account the effect of the 2:5 near commensurability between the orbital periods of Jupiter and Saturn, the secular system is characterised by ten eigen-frequencies g_i (with $g_9 = 2g_5 - g_6$ and $g_{10} = 2g_6 - g_5$) and eight eigen-frequencies f_i. We then used these in order to obtain the locations of secular resonances inside the planetary co-orbital regions[7] condensed in the six panels of Figs. 5 and 6.

Figure 5f proves that the secular resonances involving the modes g_6 and g_{10} can affect Saturn's tadpole orbits which is in agreement with recent numerical integrations by Marzari & Scholl (2000). In particular, these latter authors suggest that the mixed secular resonance involving

[7] The use of more accurate values for the eigen-frequencies from Laskar (1988) does not cause any significant alteration in the location of these secular resonances.

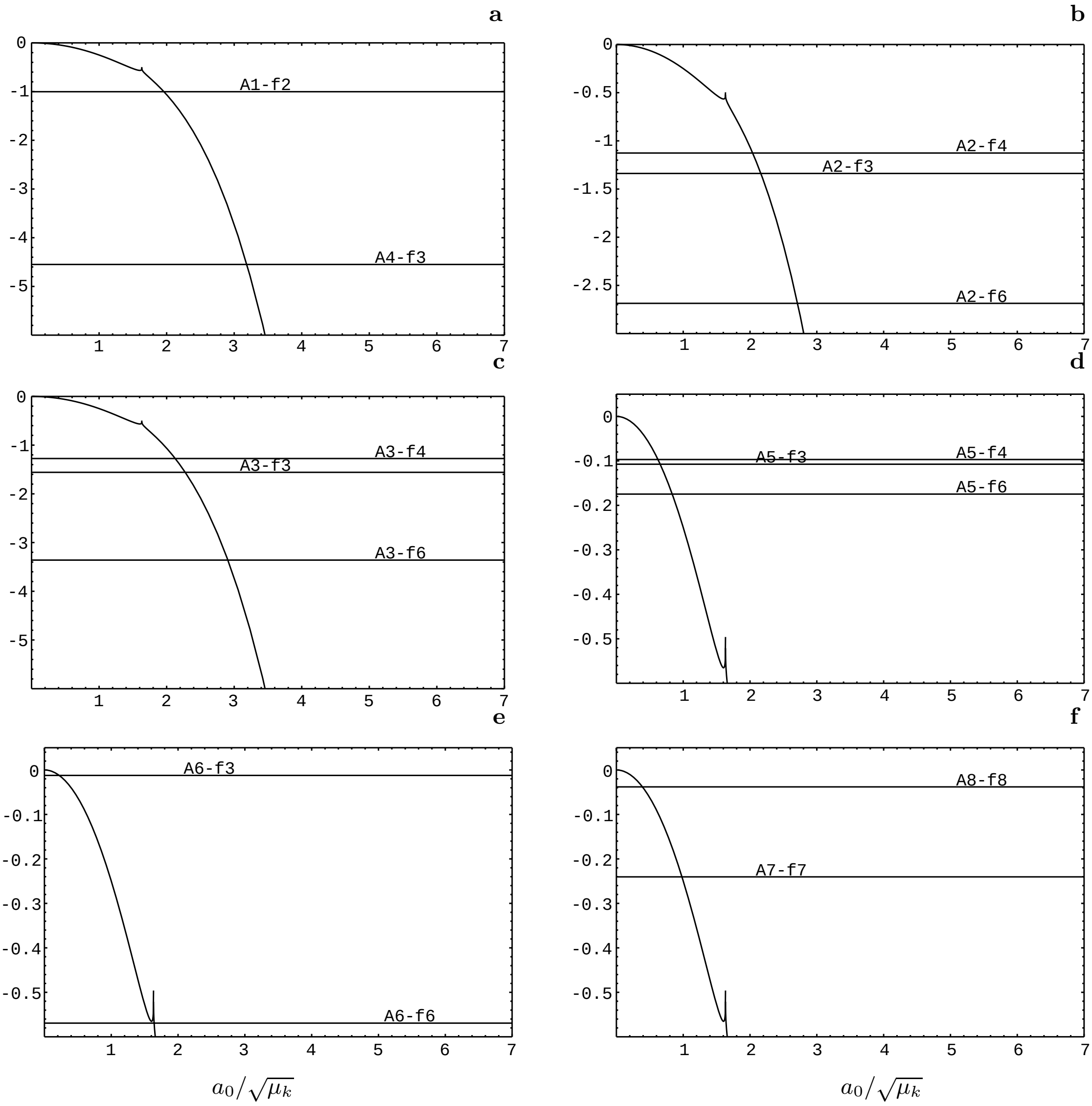

Fig. 6. Location of secular resonances involving the nodes inside the co-orbital regions of the planets. The frequencies shown in the vertical axes are adimensional and correspond to the true frequencies divided by $\mu_k n_k$. Secular resonances associated with the modes f_i occur at the locations $a_0/\sqrt{\mu_k}$ determined by the intersections of the lines labeled $B_k - f_i$ with the curves Γ_k

the mode g_{10} is the major factor responsible for the destabilization of tadpole orbits associated with Saturn.

In Fig. 6b we see that the secular resonances involving the modes f_3 and f_4 occur very close to each other inside Venus' horseshoe region, thus suggesting their possible overlap. In fact, the numerical integrations of Michel (1997) showed that a clone of asteroid (4660) Nereus becomes, at some stage in its life, a Venus' horseshoe orbit which exhibits chaotic diffusion of the inclination caused by the overlap of these two secular resonances.

From Fig. 6d and as already mentioned in Paper I, we see that the secular resonance involving the mode f_6 occurs within Jupiter's tadpole region. Recent numerical integrations by Marzari & Scholl (2000) seem to support earlier suggestions by Yoder (1979) and Milani (1994) concerning the important role played by this secular resonance in the dynamical shaping of the Trojan cloud.

5. Conclusions and discussion

In this paper we were able to construct a complete linear secular theory for Trojan-type motion (i.e. based on expansions of the disturbing potential truncated at degree two in eccentricities and inclinations). This was achieved through the use of a Hamiltonian formulation which allowed us to generalise the theory presented in Paper I by including, in a rigorous way, the effect of an oblate central mass and the secular perturbations from additional massive bodies.

Using our theory we were able to locate secular resonances inside the co-orbital regions of the uranian satellites and the planets Mercury to Neptune. Comparison with numerical integrations showed that these locations are reasonably accurate and that secular resonances seem to play a major role in determining the stability of Trojan orbits.

We end by remarking that a next necessary step would be to develop a non-linear secular theory. In particular this would allow us to obtain even more accurate locations for these secular resonances (i.e. as functions of not only the proper semi-major axis, but also the proper eccentricity and the proper inclination). Moreover, a non-linear secular theory is also essential for application to the co-orbital structures with high eccentricity and/or inclination, studied by Namouni (1999).

Acknowledgements. I thank A. Morbidelli, D. Nesvorny and the reviewer, D. Hamilton, for useful comments. This work was supported by grants PRAXIS XXI/BD/5072/95 and SFRH/BPD/1586/2000 from the Fundação para a Ciência e a Tecnologia, Portugal.

References

Brankin, R. W., Dormand, J. R., Gladwell, I., Prince, P. J., & Seward, W. L. 1987, NAG Technical Report, TR13/87
Brouwer, D., & Clemence, G. M. 1961, Methods of Celestial Mechanics (Academic Press, New York)
Brouwer, D., & Van Woerkom, A. J. J. 1950, Astron. Papers Amer. Ephem., XIII, 81
Dermott, S. F., & Murray, C. D. 1981, Icarus, 48, 1
Gladman, B. 1993, Icarus, 106, 247
Greenberg, R. 1981, AJ, 86, 912
Hénon, M., & Petit, J. M. 1986, Celest. Mech., 38, 67
Kozai, Y. 1959, AJ, 64, 367
Lagrange, J. L. 1867-1892 (Ouvres, Gauthier-Villars, Paris)
Laskar, J. 1988, A&A, 198, 341
Lichtenberg, A. J., & Lieberman, M. A. 1983, Regular and Stochastic Motion (Springer-Verlag, New York)
Malhotra, R., Fox, K., Murray, C. D., & Nicholson, P. D. 1989, A&A, 221, 348
Marzari, F., & Scholl, H. 2000, Icarus, 146, 232
Michel, P. 1997, A&A, 325, L5
Milani, A. 1994, The dynamics of the Trojan asteroids, in Asteroids Comets Meteors 1993, ed. A. Milani, et al., 159
Morais, M. H. M. 1999, A&A, 350, 318
Namouni, F. 1999, Icarus, 137, 293
Yoder, C. F. 1979, Icarus, 40, 341
Yoder, C. F., Colombo, G., Synnott, S. P., & Yoder, K. A. 1983, Icarus, 53, 431