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Chaos in pseudo-Newtonian black holes with halos

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Abstract. Newtonian as well as special relativistic dynamics are used to study the stability of orbits of a test particle moving around a black hole with a dipolar halo. The black hole is modeled by either the usual monopole potential or the Paczyńki-Wiita pseudo-Newtonian potential. The full general relativistic similar case is also considered. The Poincaré section method and the Lyapunov characteristic exponents show that the orbits for the pseudo-Newtonian potential models are more unstable than the corresponding general relativistic geodesics.

Key words. chaos – galaxies: halos – relativity – instabilities – celestial mechanics, stellar dynamics

1. Introduction

Analysis of relativistic effects in many-body simulations is not simple, due to the fact that the metric representing their gravitational interaction is far from being known. For the simplest case of two gravitating bodies the metric is known numerically only for few initial conditions and for a limited amount of time (see for example, Marronetti et al. 2000). Assuming that the metric is known, the use of the geodesic equations to determine the trajectory of the bodies is not a trivial problem.

In general, we have three main ways to consider complex systems: a) A full numeric approach with its inherent limitations due to the use of floating point arithmetics and the arbitrariness of discretizations of fundamentally continuous functions and variables. Also, we use rather unphysical ad hoc assumptions such as the introduction of numerical viscosity. b) The use of perturbative methods that are usually employed together with drastic approximations, such as the mean field approximation for the potentials in many-body simulations. These approximations introduce irreversibility in an intrinsically reversible situation. c) The modeling of the problem with simpler equations in which one takes into account a few essential features of the problem. In general, this model can be solved in a more exact form of the two preceding cases. However, we have changed the initial problem for a simpler one that may alter the results. In other words, there is not a perfect method suited to solve a complex problem. We believe that all of these methods are valid when adequate cautions are taken. Furthermore, they are complementary and the mathematical or internal consistence of the methods can be independently checked (at least for some particular cases.)

Due to the weakness of the gravitational field, far from the particles' horizon, Newtonian gravity is a reliable descriptor of the gravitational interaction. One can simulate relativistic effects within the Newtonian theory by changing the usual potentials to take into account the existence of the horizon. In other words, we can model relativistic effects using a pseudo-Newtonian potential. These models are simple enough to describe complex systems that are far beyond our current knowledge of models of full general relativity, e.g., the n-body simulation of the collision of two galaxies to any degree of resolution.

One of the simplest pseudo-Newtonian potentials used to describe the behavior of test particles moving close to a black hole is the Paczyński & Wiita (1980) pseudo-potential,

$$\Phi = -\frac{GM}{R - R_{\rm g}}. \tag{1}$$

The addition of the term $R_{\rm g} = 2GM/c^2$ critically changes the particles' trajectory near the source. Some results, such as the last stable circular orbit, are predicted in this model. Other pseudo-Newtonian models can be found in the literature, e.g., Semerák & Karas (1999), used to describe rotating black holes, i.e., to approximate the Kerr solution.

We believe that the study of the Paczyński & Wiita (PW) potential in simple, albeit nontrivial, situations may shed some light on the correctness of the pseudo-potential approach. In particular, in this article, we study integrability and chaos in a system that represents a spherically symmetric source (monopole) surrounded by a dipolar

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halo (external dipole), which is the simplest mean potential used to describe astrophysical systems restricted to a core and halo (see for instance Binney & Tremaine 1987). Different theoretical approaches are used to study this configuration. First, we use Newton's second law to find the motion equations for test particles ($\boldsymbol{a} = -\nabla\Phi$) for two different potentials that describe a core plus a dipolar halo system: a) the standard monopole plus external dipole expansion that solves the usual Laplace equation that is totally integrable, see for instance Grammaticos et al. (1985), and b) we replace in the former case the monopole term by the PW potential (1). In this case the trajectories are chaotic, as in the equivalent full general relativistic system (Vieira & Letelier 1997).

We also analyze the equivalent cases using the special relativistic dynamics. We solve the equation $a^\mu = F^\mu$ with $a^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right)$ and $F^\mu = \gamma(-\nabla\Phi \cdot v/c, -\nabla\Phi)$, where $\gamma = (1 - v^2/c^2)^{-1/2}$, and Φ is taken as in the Newtonian cases. We first use the monopole plus dipole potential that solves the Laplace equation. A phase space analysis shows that the system is stable. Replacing the monopole term by the PW potential, we obtain a very unstable system. We also review the equivalent system in general relativity. The geodesic equations for a Schwarzschild monopole plus dipolar halo give us chaotic trajectories in the phase space (Vieira & Letelier 1997).

In each of the studied cases we have an integrable Hamiltonian system of equations for the motion of a test particle moving in a spherically symmetric attraction center (standard monopole, PW potential or Schwarzschild metric) that is perturbed by an external dipole term. In all these situations we can apply the KAM (Kolmogorov, Arnold and Moser) theory, see for instance Tabor (1989). Since our mass distribution has axial symmetry, we are restricted to an effective two-dimensional problem. In the integrable case, in phase space, the orbits of test particles will be confined to a 2-torus. For a constant value of one of the coordinates, we obtain a planar section of the phase space. In the integrable case, we see closed curves for each initial condition (intersections of invariant tori). While in the non-integrable case, some tori will be destroyed and the region will be ergodically fulfilled. In order to evaluate the degree of instability of the orbits in each system we also compute the Lyapunov exponents that describe how initially close trajectories then separate.

2. Newtonian dynamics

The standard monopole plus external dipole potential in the usual cylindrical coordinates (r, z, ϕ) is

$$\Phi = -\frac{GM}{\sqrt{r^2 + z^2}} + D\,z, \tag{2}$$

where D is the dipolar strength, G the Newton constant, and M the mass of the attraction center. We use units such that $GM = 1$; furthermore, we shall take $c = 1$.

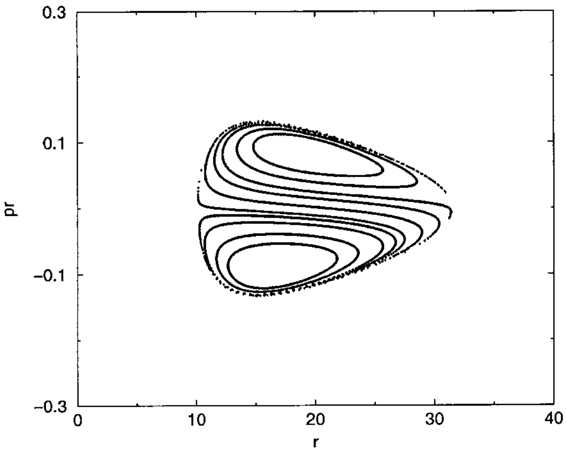

Fig. 1. Surface of section for the Newtonian motion of a test particle in a standard monopole plus external dipole potential for $L_z = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. The section corresponds to the plane $z = 0$. For these values of the parameters we have the section of an integrable motion

From the angular momentum and energy conservation we find that the motion is restricted to the region defined by

$$E^2 - 1 - \frac{L^2}{r^2} - 2\Phi \geq 0. \tag{3}$$

L is the specific angular momentum of the test particle and $E = \sqrt{1 + 2E_{\mathrm{mech}}}$, where

$$E_{\mathrm{mech}} = \frac{\dot{r}^2 + \dot{z}^2}{2} + \Phi(r, z) + \frac{L^2}{2r^2}$$

is the specific energy. Note that E becomes imaginary for $E_{\mathrm{mech}} < -0.5$, i.e. the energy of a particle located on the black hole horizon. The phase space orbits are studied using the Poincaré section method. In Fig. 1 we present the surface of section $z = 0$ for the constants: $L = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. This surface section characterizes an integrable system, as expected.

Now we shall replace the monopolar term by the PW pseudo-Newtonian potential,

$$\Phi = -\frac{1}{\sqrt{r^2 + z^2} - 2} + D\,z. \tag{4}$$

Again, the motion of test particles will be restricted to the region that solves (3) with Φ given by (4). In Fig. 2 we present the surface of section $z = 0$. We take the values for the constants as in the preceding case: $L = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. Contrary to the previous case, we observe chaotic orbits in this Poincaré section.

3. Special relativistic dynamics

In principle, the use of special relativistic dynamics should improve the modeling of general relativity with pseudo-Newtonian potentials (see Abramowicz et al. 1996). However, these authors found that the predicted spectra often differ rather substantially from those obtained in the

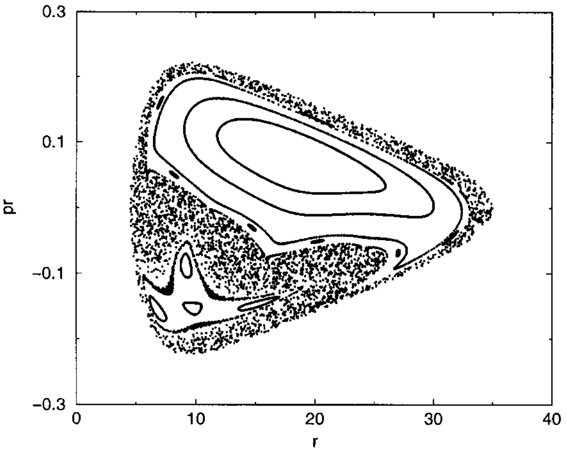

Fig. 2. Surface of section for the Newtonian motion of a test particle in a Paczyński-Wiita potential plus a dipolar halo for $L_z = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. The section corresponds to the plane $z = 0$. We see chaotic motion

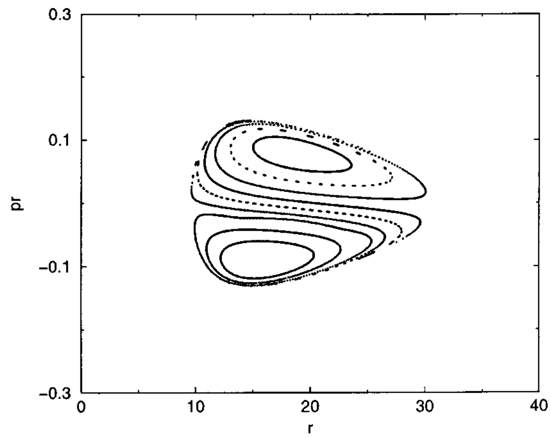

Fig. 3. Surface of section for the special relativistic motion of a test particle in a usual monopole potential plus a dipolar halo for $L_z = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. The section corresponds to the plane $z = 0$. For these values of the parameters we have the section of a regular motion

full general relativity context. From the relativistic motion equation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma + \Phi) = 0 \Rightarrow \gamma + \Phi = E, \tag{5}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{L}{\gamma\, r^2}. \tag{6}$$

By using the above equations and $u^\mu u_\mu = 1$, we obtain

$$(E - \Phi)^2 \left(1 - \dot{r}^2 - \dot{z}^2 - \frac{L^2}{[(E - \Phi)r]^2}\right) = 1\ ,$$

which is used to calculate the region in which the motion is confined. Finally, the motion equations for the variables r and z are,

$$(\Phi - E)\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\partial\Phi}{\partial r}\left(1 - \frac{\mathrm{d}r}{\mathrm{d}t}^2\right) - \frac{\partial\Phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} - \frac{L^2}{(E - \Phi)r^3}, \tag{7}$$

$$(\Phi - E)\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{\partial\Phi}{\partial z}\left(1 - \frac{\mathrm{d}z}{\mathrm{d}t}^2\right) - \frac{\partial\Phi}{\partial r}\frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t}. \tag{8}$$

As in the previous section, we start with the usual monopole plus external dipole potential field, i.e., we identify Φ with (2). In Fig. 3 we draw the Poincaré section defined by the plane $z = 0$. The constants are the same as in the preceding section, $L = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. We notice that the tori were preserved in this case, leading to stability of orbits. This is an indication of the integrability of the system.

In order to study of the PW potential plus dipolar halo, we identify Φ with (4). Unfortunately, we cannot confine the orbits by using the constants attributed to all the preceding cases. We put $L = 4.2$, $E = 0.972$, and $D = 4.2\ 10^{-4}$. Now the Poincaré section is taken as $z = -5$. The figure in this case, Fig. 4, represents a very chaotic system. We used the same constants to draw another Poincaré section for the PW potential plus dipolar halo using Newtonian dynamics. The results are presented in Fig. 5. We see some stable islands in the negative p_r region that cannot be observed Fig. 4. We conclude then that the orbits obtained in the special relativistic context are less stable than the ones obtained with Newton's law. The conjugated variables used were $\mathrm{d}r/\mathrm{d}t$ and r. We made some tests using $\mathrm{d}r/\mathrm{d}\tau$ and r, which gave results that were qualitatively the same.

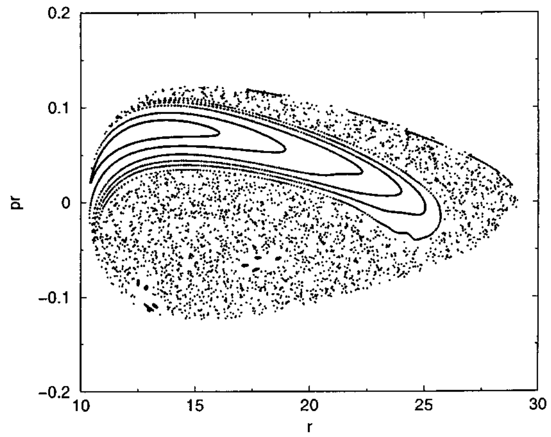

Fig. 4. Surface of section for the special relativistic motion of a test particle in a Paczyński-Wiita potential plus a dipolar halo for $L_z = 4.2$, $E = 0.972$, and $D = 4.2\ 10^{-4}$. The section corresponds to the plane $z = -5$. We have a very irregular motion

4. General relativistic dynamics

We start from the axisymmetric line element

$$\mathrm{d}s^2 = \mathrm{e}^{2\psi(u,v)}\mathrm{d}t^2 - \mathrm{e}^{-2\psi(u,v)}(u^2 - 1)(1 - v^2)\mathrm{d}\phi^2 - \mathrm{e}^{2(\gamma(u,v) - \psi(u,v))}(u^2 - v^2)\left(\frac{\mathrm{d}u^2}{u^2 - 1} + \frac{\mathrm{d}v^2}{1 - v^2}\right), \tag{9}$$

in prolate coordinates (t, u, v, ϕ). The coordinates u and v are related to the usual cylindrical coordinated by

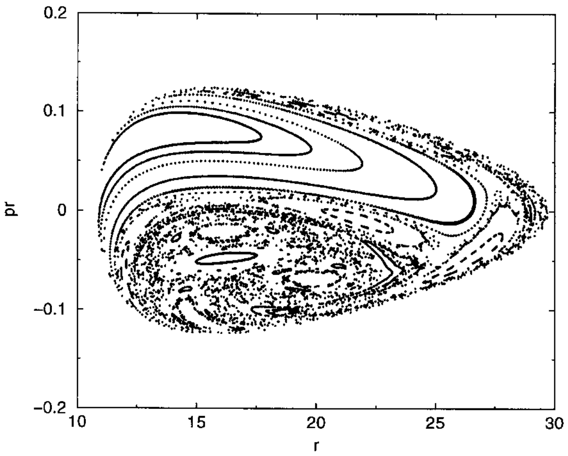

Fig. 5. Surface of section of the Newtonian motion of a test particle in a Paczyński-Wiita potential plus a dipolar halo for $L_z = 4.2$, $E = 0.972$, and $D = 4.2\ 10^{-4}$. The section corresponds to the plane $z = -5$. We have an irregular motion but it is more stable than the one shown in the preceding figure

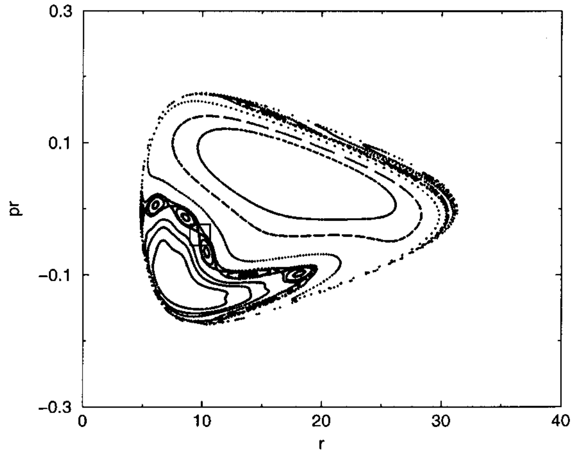

Fig. 6. Surface of section for the geodesic motion of a test particle in a Schwarzschild monopole with a dipolar halo for $L_z = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. The section corresponds to the plane $v = 0$. For these parameters we have small regions of instability

$u = (R_+ + R_-)/(2m)$ and $v = (R_+ - R_-)/(2m)$, where $R_\pm = [r^2 + (z \pm m)^2]^{1/2}$ and $m = GM/c^2$. The Schwarzschild monopole plus a dipolar halo is represented by

$$\psi(u, v) = \frac{1}{2} \log \left(\frac{1+u}{1-u} \right) + D\,uv. \tag{10}$$

Note that taking the limit, $\lim_{c^{-2}=0} \psi/c^{-2}$, with the aid of l'Hôpital rule, we recover (2). In order to obtain the appropriate units to take the limit, we need to add a c^{-2} factor to D.

The Einstein equations for this class of solutions, as well as the corresponding geodesic equations, are studied in great detail in Vieira & Letelier (1999). Due to the axial symmetry of the metric, again the effective geodesic dynamics of the test particles are restricted to a three-dimensional "phase space".

The Poincaré section is draw for $v = 0$ (which is equivalent to $z = 0$). In Fig. 6 we present the section for the values of the constants $L = 3.9$, $E = 0.976$, and $D = 2\ 10^{-4}$. Chaotic orbits can be seen in the region indicated with a rectangle. A magnification of this region is presented in Fig. 7. We can compare Fig. 6 with Fig. 2 and conclude that the orbits obtained via the geodesic equation, in general relativity are more stable than the ones obtained from the PW potential plus dipolar halo in Newtonian and special relativistic dynamics.

5. Lyapunov exponent

We study the Lyapunov exponents for the systems described above to better analyze the stability of the orbits. We use the Lyapunov characteristic number (LCN) that is defined as the double limit

$$LCN = \lim_{\substack{\delta_0 \to 0 \\ t \to \infty}} \left[\frac{\log(\delta/\delta_0)}{t} \right], \tag{11}$$

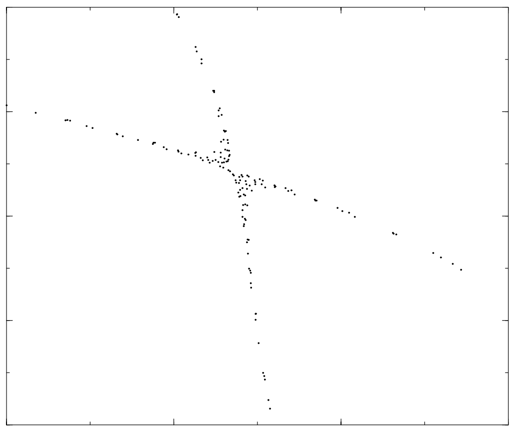

Fig. 7. A magnification of the small boxed region of the previous figure

where δ_0 and δ are the deviation of two nearby orbits at times 0 and t respectively. We get the largest LCN using the technique suggested by Benettin et al. (1976)

We begin by comparing the LCN for orbits in a PW+Dipole system in special relativity and the LCN for orbits in a PW+Dipole in Newtonian theory. The constants are $L = 4.1$, $E = 0.972$, and $D = 4.1\ 10^{-4}$. The maximum LCN was obtained around $r = 20$, $z = -5$, and $p_r = -0.04$. Note that the value of p_z is determined by the constants of motion and the value of r, z and p_r. For the relativistic case we get $LCN = (3.2 \pm 0.4)\ 10^{-4}$ while for the Newtonian approach we obtain $LCN = (1.8 \pm 0.4)\ 10^{-4}$. We tested the usual integrable Newtonian monopole plus dipole system and we always obtain a LCN at least one order of magnitude lower that the preceding case.

For orbits of test particles in the the full general relativistic monopole plus dipole system and in the Newtonian PW+Dipole system we chose $L = 3.902$, $E = 0.9756$ and $D = 2.0\ 10^{-4}$. We obtain for orbits in the PW+Dipole system $LCN = (2.0 \pm 0.5)\ 10^{-4}$. This value was obtained for orbits around $r = 7.5, z = 0$, and $p_r = 0$. For the general relativistic system the proper time and the coordinate time were tested in Eq. (11) and no significant difference was found. The largest LCN was computed around $u = 9.75, v = 0$, and $p_u = -0.038$. As before, p_v is fixed by the value of the other variables and the motion constants. We always found $LCN < 5\ 10^{-5}$. The Lyapunov-like coefficients used in general relativistic systems may have different forms, such as the one suggested by Burd & Tavakol (1993) in the study of Bianchi IX systems. However, we have studied a simple system with no singularities other than the black hole, where we have a well-defined evolution parameter. Hence, in this case, no significant difference should be found by using other definitions of the Lyapunov coefficients. Furthermore, in the general relativistic system studied we have several natural ways to choose the space variables e.g., the spheroidal (u, v, ψ) and the cylindrical (r, z, ϕ). We found no significant differences when either system of coordinates were used to describe the orbits of particles moving a few Schwarzschild radii away from the central black hole.

In summary, the study of Lyapunov coefficients confirms the qualitative analysis of the Poincaré section method, i.e. that the general relativistic orbits are more stable than the Newtonian and special relativistic ones. The special relativistic orbits are the most unstable.

6. Discussion

In the Paczyński-Witta potential, the term $-2GM/c^2$ in the denominator of Eq. (4) creates a saddle point in the effective potential in Newtonian as well as in special relativistic dynamics. The addition of the dipole term separates the stable and unstable manifold emanating from the hyperbolic fixed point, as discussed by Letelier & Vieira (1998). In this case, as a consequence of the Poincaré-Birkhoff theorem, there is an homoclinic web that gives rise to chaotic motion for bounded orbits in phase space, see for instance Tabor (1989).

The chaotic orbits encountered in the pseudo-Newtonian plus dipole system agrees with the general relativistic equivalent situation. However, those effects might be distorted in the PW approach because the Poincaré sections as well as the Lyapunov exponents show more unstable orbits. This instability is magnified when special relativistic dynamics is used. Vokrouhlický & Karas (1998) studied the stability of orbits for particles gravitating around a $1/R$ Newtonian potential with an axisymmetric perturbation. Sridhar & Touma (1999) found for the same class of potentials that the instability decreases in orbits closer to the black hole. This result may not be verified when pseudo-Newtonian or full general relativistic models are considered, the main difference being the presence of a saddle point in the effective potential near the black hole. Therefore, orbits near the core may be more unstable because of this critical point in the effective potential that is a source of instability.

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