A&A 368, 325–334 (2001)
DOI: 10.1051/0004-6361:20000525

Astronomy & Astrophysics

Radiating gravitational collapse with shearing motion and bulk viscosity

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Received 15 November 2000 / Accepted 14 December 2000

Abstract. A model is proposed of a collapsing radiating star consisting of a shearing fluid with bulk viscosity undergoing radial heat flow with outgoing radiation. The pressure of the star, at the beginning of the collapse, is isotropic but due to the presence of the bulk viscosity the pressure becomes more and more anisotropic. The behavior of the density, pressure, mass, luminosity, the effective adiabatic index and the Kretschmann scalar is analyzed. Our work is compared to the case of a collapsing shearing fluid of a previous model, for a star with 6 $M_{\odot}$.

Key words. black hole physics – dense matter – gravitation – relativity – radiation mechanism, thermal

1. Introduction

The majority of the previous works in gravitational collapse have considered only shear-free motion of the fluid (de Oliveira et al. 1985; Bonnor et al. 1989). This simplification allows us to obtain exact solutions of the Einstein's equations in some cases but it is somewhat unrealistic. It is also unrealistic to consider heat flow without viscosity but if viscosity is introduced, it is desirable to allow shear in the fluid motion. In a recent work Martínez & Pavón (1994) have studied the collapse of a radiating star with bulk viscosity but they still maintained the shear-free motion of the fluid. Thus, it is interesting to study solutions that contains shear (and bulk viscosity), because it plays a very important role in the study of gravitational collapse, as shown in Chan (1997, 1998, 2000).

In the first paper (Chan 1997, 1998) we have compared two collapsing model: a shear-free and a shearing model. We were interested in studying the effect of the shearing motion in the evolution of the collapse. It was shown that the pressure of the star, at the beginning of the collapse, is isotropic but due to the presence of the shear the pressure becomes more and more anisotropic. The anisotropy in self-gravitating systems has been reviewed and discussed the causes for its appearance in Herrera & Santos (1997). As shown by Chan (1997, 1998) the simplest cause of the presence of anisotropy in a self-gravitating body is the shearing motion of the fluid, because it appears without an imposition ad-hoc (Chan 1993).

In the second paper (Chan 2000) we have analyzed a model of a collapsing radiating star consisting of an isotropic fluid with shear viscosity undergoing radial heat flow with outgoing radiation, but without bulk viscosity.

The aim of this work is to generalize our previous model by introducing bulk viscosity, besides the shear motion of the fluid, and compare it to the non-viscous collapse.

This work is organized as follows. In Sect. 2 we present the Einstein's field equations. In Sect. 3 we rederive the junction conditions, since Chan (1997, 1998, 2000) have obtained only results without bulk viscosity. In Sect. 4 we present the proposed solution of the field equations. In Sect. 5 we describe the model considered in this work for the initial configuration. In Sect. 6 we present the energy conditions for a viscous anisotropic fluid. In Sect. 7 we show the time evolution of the total mass, luminosity and the effective adiabatic index and in Sect. 8 we summarize the main results obtained in this work.

2. Field equations

We assume a spherically symmetric distribution of fluid undergoing dissipation in the form of heat flow. While the dissipative fluid collapses it produces radiation. The interior spacetime is described by the most general spherically symmetric metric, using comoving coordinates,

$$\begin{aligned} ds_-^2 \;=\; & -A^2(r,t)dt^2 + B^2(r,t)dr^2 \\ & +C^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \quad (1)$$

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The exterior spacetime is described by Vaidya's (1953) metric, which represents an outgoing radial flux of radiation,

$$\mathrm{d}s_+^2 = -\left[1 - \frac{2m(v)}{\boldsymbol{r}}\right]\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}\boldsymbol{r} + \boldsymbol{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \quad (2)$$

where $m(v)$ represents the mass of the system inside the boundary surface Σ, function of the retarded time v.

We assume the interior energy-momentum tensor is given by

$$G_{\alpha\beta} = \kappa T_{\alpha\beta} = \kappa\left[(\mu + p_\mathrm{t})u_\alpha u_\beta + p_\mathrm{t} g_{\alpha\beta} + (p - p_\mathrm{t})X_\alpha X_\beta + q_\alpha u_\beta + q_\beta u_\alpha - \zeta\Theta(g_{\alpha\beta} + u_\alpha u_\beta)\right], \quad (3)$$

where μ is the energy density of the fluid, p is the radial pressure, p_t is the tangential pressure, q^α is the radial heat flux, X_α is an unit four-vector along the radial direction, u^α is the four-velocity, which have to satisfy $u^\alpha q_\alpha = 0$, $X_\alpha X^\alpha = 1$, $X_\alpha u^\alpha = 0$ and $\kappa = 8\pi$ (i.e., $c = G = 1$). The quantity $\zeta > 0$ is the coefficient of bulk viscosity and the expansion scalar Θ is defined as

$$\Theta = u^\alpha_{;\alpha}, \quad (4)$$

where the semicolon denotes a covariant derivative.

Since we utilize comoving coordinates we have,

$$u^\alpha = A^{-1}\delta^\alpha_0, \quad (5)$$

and since the heat flux is radial

$$q^\alpha = q\delta^\alpha_1. \quad (6)$$

Using (1) and (4), we can write that

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}\right). \quad (7)$$

The non-vanishing components of the field equations, using (1), (3), (5), (6) and (4), interior of the boundary surface Σ are

$$G^-_{00} = -\left(\frac{A}{B}\right)^2\left[2\frac{C''}{C} + \left(\frac{C'}{C}\right)^2 - 2\frac{C'}{C}\frac{B'}{B}\right] + \left(\frac{A}{C}\right)^2 + \frac{\dot{C}}{C}\left(\frac{\dot{C}}{C} + 2\frac{\dot{B}}{B}\right) = \kappa A^2\mu, \quad (8)$$

$$G^-_{11} = \frac{C'}{C}\left(\frac{C'}{C} + 2\frac{A'}{A}\right) - \left(\frac{B}{C}\right)^2 - \left(\frac{B}{A}\right)^2\left[2\frac{\ddot{C}}{C} + \left(\frac{\dot{C}}{C}\right)^2 - 2\frac{\dot{A}}{A}\frac{\dot{C}}{C}\right] = \kappa B^2(p - \zeta\Theta), \quad (9)$$

$$G^-_{22} = \left(\frac{C}{B}\right)^2\left[\frac{C''}{C} + \frac{A''}{A} + \frac{C'}{C}\frac{A'}{A} - \frac{A'}{A}\frac{B'}{B} - \frac{B'}{B}\frac{C'}{C}\right] + \left(\frac{C}{A}\right)^2\left[-\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{C}}{C}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{B}}{B}\right] = \kappa C^2(p_\mathrm{t} - \zeta\Theta), \quad (10)$$

$$G^-_{33} = G^-_{22}\sin^2\theta, \quad (11)$$

$$G^-_{01} = -2\frac{\dot{C}'}{C} + 2\frac{C'}{C}\frac{\dot{B}}{B} + 2\frac{A'}{A}\frac{\dot{C}}{C} = -\kappa AB^2 q. \quad (12)$$

The dot and the prime stand for differentiation with respect to t and r, respectively.

3. Junction conditions

We consider a spherical surface with its motion described by a time-like three-space Σ, which divides spacetimes into interior and exterior manifolds. For the junction conditions we follow the approach given by Israel (1966a, israel-66b). Hence we have to demand

$$(\mathrm{d}s^2_-)_\Sigma = (\mathrm{d}s^2_+)_\Sigma, \quad (13)$$

$$K^-_{ij} = K^+_{ij}, \quad (14)$$

where $K^\pm_{ij}$ is the extrinsic curvature to Σ, given by

$$K^\pm_{ij} = -n^\pm_\alpha\frac{\partial^2 x^\alpha}{\partial\xi^i\partial\xi^j} - n^\pm_\alpha\Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial\xi^i}\frac{\partial x^\gamma}{\partial\xi^j}, \quad (15)$$

and where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, $n^\pm_\alpha$ the unit normal vectors to Σ, x^α are the coordinates of interior and exterior spacetimes and ξ^i are the coordinates that define the surface Σ.

From the junction condition (13) we obtain

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = A(r_\Sigma, t)^{-1}, \quad (16)$$

$$C(r_\Sigma, t) = \boldsymbol{r}_\Sigma(v), \quad (17)$$

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^{-2}_\Sigma = \left(1 - \frac{2m}{\boldsymbol{r}} + 2\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}v}\right)_\Sigma, \quad (18)$$

where τ is a time coordinate defined only on Σ.

The unit normal vectors to Σ (for details see Santos 1985) are given by

$$n^-_\alpha = B(r_\Sigma, t)\delta^1_\alpha, \quad (19)$$

$$n^+_\alpha = \left(1 - \frac{2m}{\boldsymbol{r}} + 2\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}v}\right)^{-1/2}_\Sigma\left(-\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}v}\delta^0_\alpha + \delta^1_\alpha\right)_\Sigma. \quad (20)$$

The non-vanishing extrinsic curvature components are given by

$$K^-_{\tau\tau} = -\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\frac{A'A}{B}\right]_\Sigma, \quad (21)$$

$$K^-_{\theta\theta} = \left(\frac{C'C}{B}\right)_\Sigma, \quad (22)$$

$$K^-_{\phi\phi} = K^-_{\theta\theta}\sin^2\theta, \quad (23)$$

$$K^+_{\tau\tau} = \left[\frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^{-1} - \left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)\frac{m}{\boldsymbol{r}^2}\right]_\Sigma, \quad (24)$$

$$K^+_{\theta\theta} = \left[\left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right) \left(1 - \frac{2m}{\boldsymbol{r}} \right) \boldsymbol{r} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} \boldsymbol{r} \right]_\Sigma, \tag{25}$$

$$K^+_{\phi\phi} = K^+_{\theta\theta} \sin^2\theta. \tag{26}$$

From Eqs. (22) and (25) we have

$$\left[\left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right) \left(1 - \frac{2m}{\boldsymbol{r}} \right) \boldsymbol{r} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} \boldsymbol{r} \right]_\Sigma = \left(\frac{C'C}{B} \right)_\Sigma. \tag{27}$$

With the help of Eqs. (16), (17), (18), we can write (27) as

$$m = \left\{ \frac{C}{2} \left[1 + \left(\frac{\dot{C}}{A} \right)^2 - \left(\frac{C'}{B} \right)^2 \right] \right\}_\Sigma, \tag{28}$$

which is the total energy entrapped inside the surface Σ (Cahill & Mcvittie 1970).

From the Eqs. (21) and (24), using (16), we have

$$\left[\frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2} \left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right)^{-1} - \left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right) \frac{m}{\boldsymbol{r}^2} \right]_\Sigma = - \left(\frac{A'}{AB} \right)_\Sigma. \tag{29}$$

Substituting Eqs. (16), (17) and (28) into (27) we can write

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right)_\Sigma = \left(\frac{C'}{B} + \frac{\dot{C}}{A} \right)^{-1}_\Sigma. \tag{30}$$

Differentiating (30) with respect to τ and using Eqs. (28), (30), we can rewrite (29) as

$$\left(\frac{C}{2AB} \right)_\Sigma \left\{ 2\frac{\dot{C}'}{C} - 2\frac{C'}{C}\frac{\dot{B}}{B} - 2\frac{A'}{A}\frac{\dot{C}}{C} + \left(\frac{B}{A} \right) \left[2\frac{\ddot{C}}{C} - 2\frac{\dot{C}}{C}\frac{\dot{A}}{A} + \left(\frac{A}{C} \right)^2 + \left(\frac{\dot{C}}{C} \right)^2 - \left(\frac{A}{B} \right)^2 \left(\frac{C'}{C} \right)^2 - \left(\frac{A}{B} \right)^2 \left(2\frac{A'}{A}\frac{C'}{C} \right) \right] \right\}_\Sigma = 0. \tag{31}$$

Comparing (31) with (9) and (12), we can finally write

$$(p - \zeta\Theta)_\Sigma = (qB)_\Sigma. \tag{32}$$

This result is analogous to the one obtained by Chan (1997, 1998, 2000) for a shearing fluid motion but now we have generalized for an interior fluid with bulk viscosity.

The total luminosity for an observer at rest at infinity is

$$L_\infty = - \left(\frac{\mathrm{d}m}{\mathrm{d}v} \right)_\Sigma = - \left[\frac{\mathrm{d}m}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}\tau} \left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right)^{-1} \right]_\Sigma. \tag{33}$$

Differentiating (28) with respect to t, using (16), (30) and (9), we obtain that

$$L_\infty = \frac{\kappa}{2} \left[(p - \zeta\Theta) C^2 \left(\frac{C'}{B} + \frac{\dot{C}}{A} \right)^2 \right]_\Sigma. \tag{34}$$

The boundary redshift can be used to determine the time of formation of the horizon. The boundary redshift z_Σ is given by

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\tau} \right)_\Sigma = 1 + z_\Sigma. \tag{35}$$

The redshift, for an observer at rest at infinity diverges at the time of formation of the black hole. From (30) we can see that this happens when

$$\left(\frac{C'}{B} + \frac{\dot{C}}{A} \right)_\Sigma = 0. \tag{36}$$

4. Solution of the field equations

Again as in Chan (1997, 1998, 2000) we propose solutions of the field Eqs. (8)–(12) with the form

$$A(r,t) = A_0(r), \tag{37}$$

$$B(r,t) = B_0(r), \tag{38}$$

$$C(r,t) = rB_0(r)f(t), \tag{39}$$

where $A_0(r)$ and $B_0(r)$ are solutions of a static perfect fluid having μ_0 as the energy density and p_0 as the isotropic pressure. We remark that, following the junction condition Eq. (17), the function $C(r_\Sigma, t)$ represents the luminosity radius of the body as seen by an exterior observer.

Thus, the expansion scalar (7) can be written as

$$\Theta = \frac{2}{A_0}\frac{\dot{f}}{f}. \tag{40}$$

Now Eqs. (8)–(12) can be written as

$$\kappa\mu = \kappa\mu_0 + \frac{1}{A_0^2} \left(\frac{\dot{f}}{f} \right)^2 + \frac{1}{r^2 B_0^2} \left(\frac{1}{f^2} - 1 \right), \tag{41}$$

$$\kappa p = \kappa p_0 - \frac{1}{A_0^2} \left[2\frac{\ddot{f}}{f} + \left(\frac{\dot{f}}{f} \right)^2 \right] - \frac{1}{r^2 B_0^2} \left(\frac{1}{f^2} - 1 \right) + \frac{2\kappa\zeta}{A_0}\frac{\dot{f}}{f}, \tag{42}$$

$$\kappa p_t = \kappa p_0 - \frac{1}{A_0^2}\frac{\ddot{f}}{f} + \frac{2\kappa\zeta}{A_0}\frac{\dot{f}}{f}, \tag{43}$$

$$\kappa q = \frac{2}{A_0 B_0^2} \left(\frac{\dot{f}}{f} \right) \left(\frac{B_0'}{B_0} + \frac{1}{r} - \frac{A_0'}{A_0} \right), \tag{44}$$

where

$$\kappa\mu_0 = -\frac{1}{B_0^2} \left[2\frac{B_0''}{B_0} - \left(\frac{B_0'}{B_0} \right)^2 + \frac{4}{r}\frac{B_0'}{B_0} \right], \tag{45}$$

$$\kappa p_0 = \frac{1}{B_0^2} \left[\left(\frac{B_0'}{B_0} \right)^2 + \frac{2}{r}\frac{B_0'}{B_0} + 2\frac{A_0'}{A_0}\frac{B_0'}{B_0} + \frac{2}{r}\frac{A_0'}{A_0} \right]. \tag{46}$$

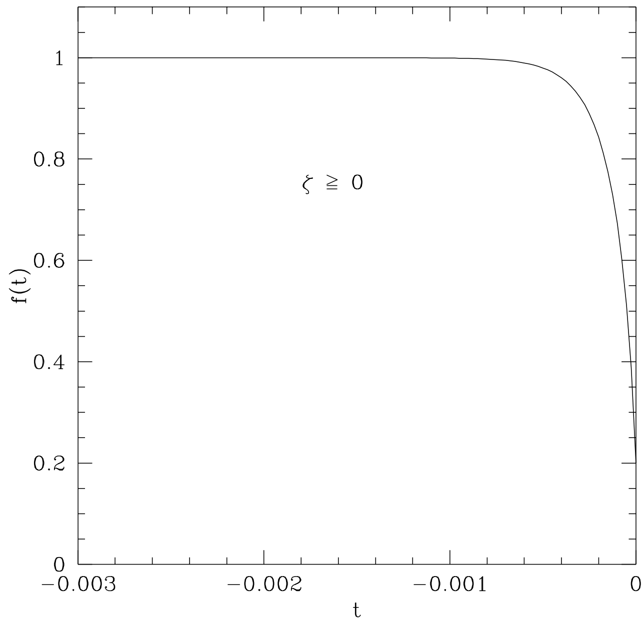

Fig. 1. Time behavior of the function $f(t)$ for the models with and without bulk viscosity. The time is in units of second and $f(t)$ is dimensionless

We can see from Eqs. (41)–(44) that when the function $f(t) = 1$ we obtain the static perfect fluid configuration.

Substituting Eqs. (42), (44) and (40) into (32), assuming also that $p_0(r_\Sigma) = 0$, we obtain a second order differential equation in $f(t)$,

$$2f\ddot{f} + \dot{f}^2 + af\dot{f} + b(1 - f^2) = 0, \tag{47}$$

where

$$a = \left[2\left(\frac{A_0}{B_0}\right)\left(\frac{B_0'}{B_0} + \frac{1}{r} - \frac{A_0'}{A_0}\right)\right]_\Sigma, \tag{48}$$

and

$$b = \left(\frac{A_0^2}{r^2 B_0^2}\right)_\Sigma. \tag{49}$$

This equation is identical to the one obtained in Chan (1997, 1998, 2000). Thus, as before it has to be solved numerically (Fig. 1), assuming that at $t \to -\infty$ represents the static configuration with $\dot{f}(t \to -\infty) \to 0$ and $f(t \to -\infty) \to 1$. We also assume that $f(t \to 0) \to 0$. This means that the luminosity radius $C(r_\Sigma, t)$ has the value $r_\Sigma B_0(r_\Sigma)$ at the beginning of the collapse and vanishing at the end of the evolution.

5. Model of the initial configuration

We consider that the system at the beginning of the collapse has a static configuration of a perfect fluid satisfying the Schwarzschild interior solution (Raychaudhuri & Maiti 1979)

$$A_0 = \frac{g(r)}{2(1 + r_\Sigma^2)(1 + r^2)}, \tag{50}$$

$$B_0 = \frac{2R}{1 + r^2}, \tag{51}$$

where

$$g(r) = 3(1 - r_\Sigma^2)(1 + r^2) - (1 + r_\Sigma^2)(1 - r^2), \tag{52}$$

and

$$R = m_0 \frac{(1 + r_\Sigma^2)^3}{4r_\Sigma^3}. \tag{53}$$

and where r_Σ is the initial radius of the star in comoving coordinates and m_0 is the initial mass of the system. Thus the static uniform energy density and static pressure are given by

$$\kappa\mu_0 = \frac{3}{R^2}, \tag{54}$$

$$\kappa p_0 = \frac{6}{R^2}\frac{(r_\Sigma^2 - r^2)}{g(r)}. \tag{55}$$

We consider the initial configuration as due to a helium core of a presupernova with $m_0 = 6\ M_\odot$, initial radius $r_\Sigma = 1.6\ 10^5$ km (Woosley & Phillips 1988). With these values we can solve numerically the differential Eq. (47). We can see from (44), using (50)–(53) and this initial configuration, that $[(B_0'/B_0 + 1/r - A_0'/A_0)/A_0]_\Sigma < 0$ and by the fact that $q_\Sigma > 0$ then we conclude that $\dot{f} < 0$.

In order to determine the time of formation of the horizon $f_{\rm bh}$, we use Eqs. (28), (36) and (37)–(39) and write

$$\frac{\dot{f}_{\rm bh}}{f_{\rm bh}} = -\left[\frac{A_0}{B_0}\left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\right]_\Sigma \approx -3.606\ 10^3, \tag{56}$$

which gives us $f_{\rm bh} \approx 0.673$ ($t \approx -1.275\ 10^{-4}$), using the numerical solution of $f(t)$.

We will assume that ζ is constant, but in general the bulk viscosity coefficient depends on the temperature and density of the fluid (Cutler & Lindblom 1987). Hereinafter, the values of ζ will be $1.347\ 10^{30}$, $6.736\ 10^{30}$ and $1.347\ 10^{31}$ g cm^{-1} s^{-1}, which correspond to values 100, 500 and 1000 s^{-1}, respectively, in time units.

It is shown in Figs. 2 and 6 the radial profiles of the density and the heat flux. It is shown only one plot for each quantity because they do not depend on the bulk viscosity, which can be seen from Eqs. (41) and (44).

In Figs. 3 and 4 we plot the radial profiles of the radial and tangential pressures. In these figures we notice that the radial and tangential pressure diminish with the viscosity.

In Fig. 5 is shown the radial profiles of the radial and tangential pressure ratio. In this figure ($\zeta = 0$) we can see that the star is isotropic at the beginning of the collapse ($f = 1$) but becoming more and more anisotropic at later times. The anisotropy for the viscous model ($\zeta \neq 0$) has the same time behavior, except for $\zeta = 1000$.

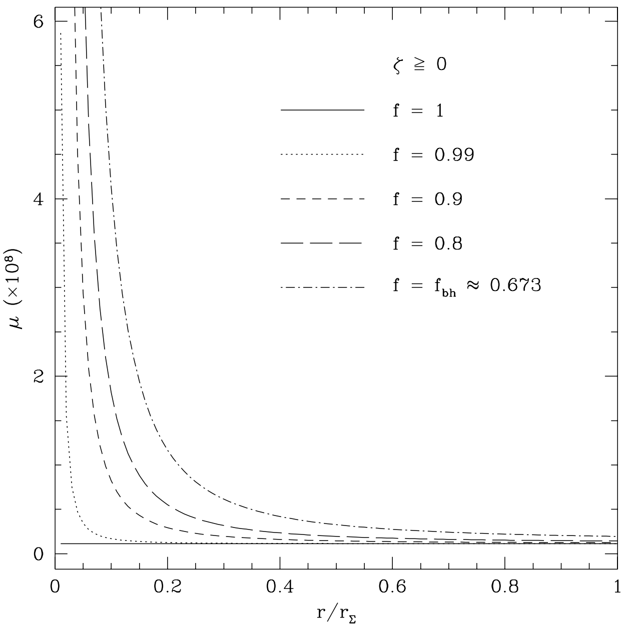

Fig. 2. Density profiles for the model with and without bulk viscosity. The radii r and r_Σ are in units of seconds and the density is in units of s^{-2}

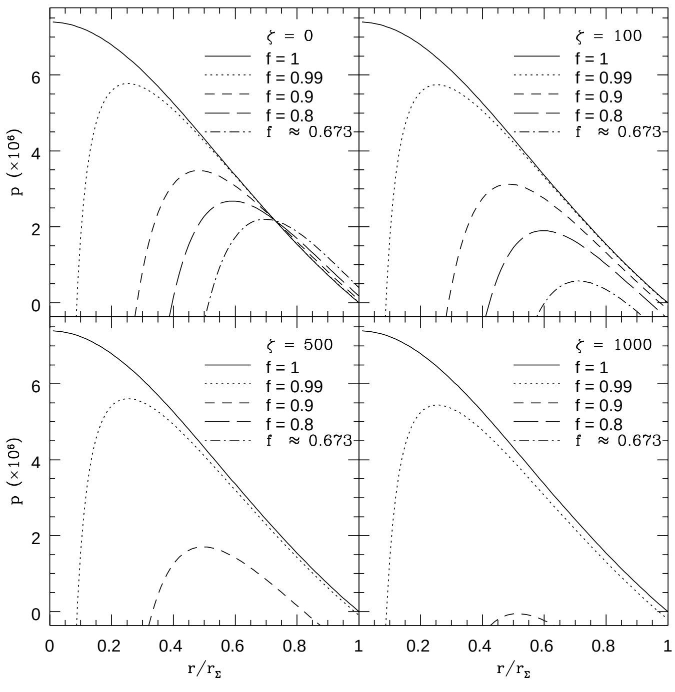

Fig. 3. Radial pressure profiles for four different values of ζ. The radii r and r_Σ are in units of seconds and the radial pressure is in units of s^{-2}

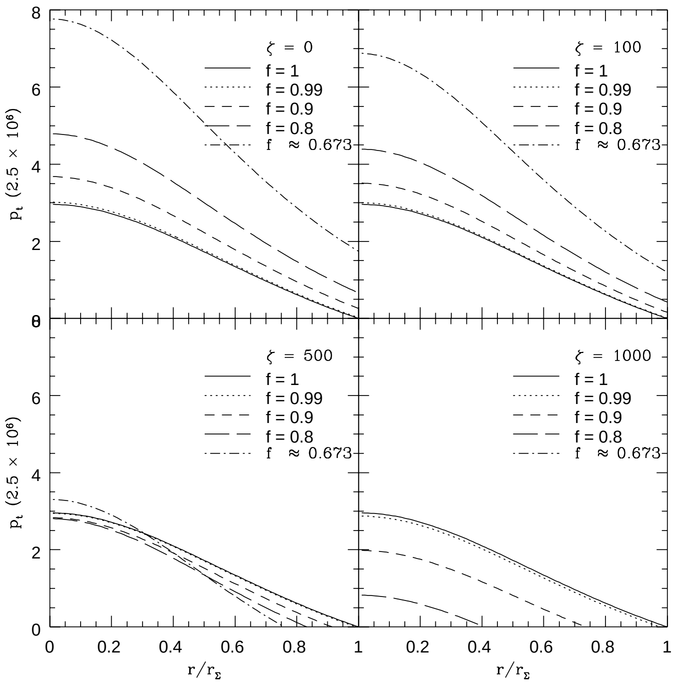

Fig. 4. Tangential pressure profiles for four different values of ζ. The radii r and r_Σ are in units of seconds and the tangential pressure p_t is in units of s^{-2}

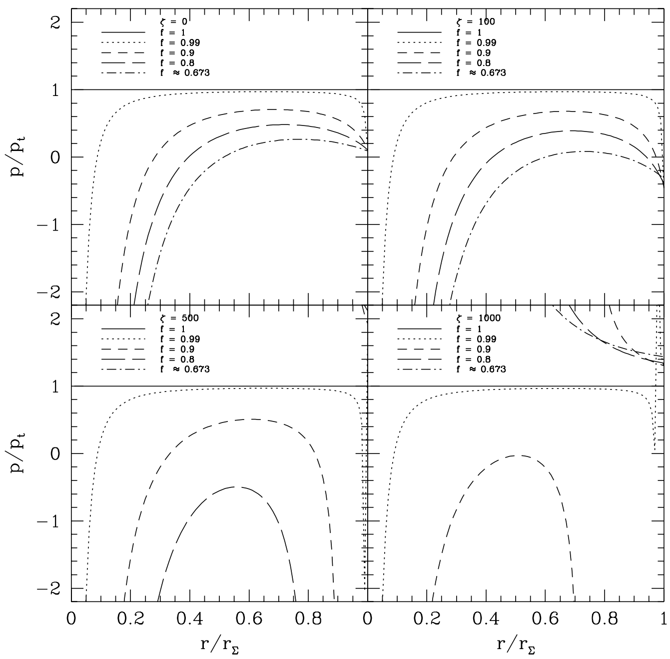

Fig. 5. The profiles for four different values of ζ of the ratio between the radial and tangential pressures. The radii r and r_Σ are in units of seconds; and the radial and tangential pressure, p and p_t, are in units of s^{-2}

6. Energy conditions for a bulk viscous anisotropic fluid

All known forms of matter obey the weak, dominant and strong energy conditions. For this reason a star model based on some fluid which violates these conditions cannot be seriously considered as physically relevant. Thus, in order to find the energy conditions, we have followed the same procedure used in Kolassis et al. (1988) and have generalized the energy conditions for a viscous anisotropic fluid.

For the energy-momentum tensor Segre type $[111, 1]$ and if λ_0 denotes the eigenvalue corresponding to the timelike eigenvector, the general energy conditions are

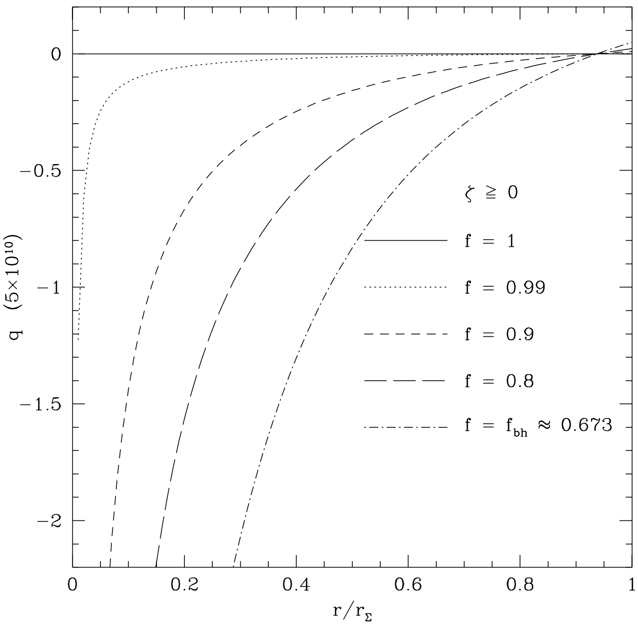

Fig. 6. Heat flux scalar profiles for the model with and without bulk viscosity. The radius r and r_Σ are in units of seconds and the heat flux q is in units of s^{-2}

equivalent to the following relations between the eigenvalues of the energy-momentum tensor:

a) weak energy condition

$$-\lambda_0 \geq 0, \tag{57}$$

and

$$-\lambda_0 + \lambda_i \geq 0, \tag{58}$$

b) dominant energy condition

$$\lambda_0 \leq \lambda_i \leq -\lambda_0, \tag{59}$$

c) strong energy condition

$$-\lambda_0 + \sum_i \lambda_i \geq 0, \tag{60}$$

and

$$-\lambda_0 + \lambda_i \geq 0, \tag{61}$$

where the values $i = 1, 2, 3$ represent the eigenvalues corresponding to the spacelike eigenvectors.

The eigenvalues λ of the energy-momentum tensor are the roots of the equation

$$|T_{\alpha\beta} - \lambda g_{\alpha\beta}| = 0. \tag{62}$$

Since λ is a scalar we can use a locally Minkowskian coordinate system, we have

$$u^\alpha = \delta_0^\alpha, \tag{63}$$

$$X^\alpha = \delta_1^\alpha, \tag{64}$$

$$q^\alpha = q\delta_1^\alpha. \tag{65}$$

Thus, we can rewrite Eq. (62) as

$$\begin{vmatrix} \mu+\lambda & -q & 0 & 0 \\ -q & p-\lambda-\zeta\Theta & 0 & 0 \\ 0 & 0 & p_t-\lambda-\zeta\Theta & 0 \\ 0 & 0 & 0 & p_t-\lambda-\zeta\Theta \end{vmatrix} = 0,$$

where the determinant of this equation is given by

$$\left[(\mu+\lambda)(\lambda - p + \zeta\Theta) + q^2\right](\lambda - p_t + \zeta\Theta)^2 = 0. \tag{66}$$

Thus, one of the solutions of Eq. (66) is

$$\left[(\mu+\lambda)(\lambda - p + \zeta\Theta) + q^2\right] = 0, \tag{67}$$

which can be rewritten as

$$\lambda^2 + (\mu - p + \zeta\Theta)\lambda + q^2 - \mu(p - \zeta\Theta) = 0. \tag{68}$$

The two roots of Eq. (68) are

$$\lambda_0 = -\frac{1}{2}(\mu - p + \zeta\Theta + \Delta), \tag{69}$$

and

$$\lambda_1 = -\frac{1}{2}(\mu - p + \zeta\Theta - \Delta), \tag{70}$$

where

$$\Delta^2 = (\mu + p - \zeta\Theta)^2 - 4q^2 \geq 0, \tag{71}$$

must be greater or equal to zero in order to have real solutions. This equation can be rewritten as

$$|\mu + p - \zeta\Theta| - 2|q| \geq 0. \tag{72}$$

The second solution of Eq. (66) is

$$(\lambda - p_t + \zeta\Theta)^2 = 0, \tag{73}$$

whose roots are given by

$$\lambda_2 = p_t - \zeta\Theta, \tag{74}$$

and

$$\lambda_3 = p_t - \zeta\Theta. \tag{75}$$

6.1. Weak energy conditions

From Eqs. (57) and (69) we get the first weak energy condition written as

$$\mu - p + \zeta\Theta + \Delta \geq 0. \tag{76}$$

From Eq. (58), setting $i = 1$ and using Eqs. (69) and (70) we get the second weak energy condition given by

$$\Delta \geq 0, \tag{77}$$

which is equal to the condition (71).

From Eq. (58), now setting $i = 2, 3$ (since λ_2 and λ_3 are identical) and using Eqs. (69) and (74)–(75) we get the third weak energy condition given by

$$\mu - p + 2p_t - \zeta\Theta + \Delta \geq 0. \tag{78}$$

6.2. Dominant energy conditions

From Eq. (59), setting $i = 1$ and using Eqs. (69) and (70) we get the inequality

$$-(\mu - p + \zeta\Theta + \Delta) \le -(\mu - p + \zeta\Theta - \Delta) \le \mu - p + \zeta\Theta + \Delta, \quad (79)$$

which can be split into two inequalities, given by

$$\Delta \ge 0, \quad (80)$$

and

$$\mu - p + \zeta\Theta \ge 0. \quad (81)$$

From Eq. (59), setting $i = 2, 3$ (again because λ_2 and λ_3 are identical) and using Eqs. (69) and (74)–(75) we get the inequality

$$-(\mu - p + \zeta\Theta + \Delta) \le 2(p_t - \zeta\Theta) \le \mu - p + \zeta\Theta + \Delta, \quad (82)$$

which again we can split it into two inequalities, given by

$$\mu - p + 2p_t - \zeta\Theta + \Delta \ge 0, \quad (83)$$

and

$$\mu - p - 2p_t + 3\zeta\Theta + \Delta \ge 0. \quad (84)$$

6.3. Strong energy conditions

Substituting Eqs. (69), (70), (74) and (75) into Eq. (60) we get the first strong energy condition given by

$$2p_t - 2\zeta\Theta + \Delta \ge 0. \quad (85)$$

Since one of the weak energy conditions, Eq. (58), is the same for the strong energy condition (Eq. (61)), thus we have that the second and third strong energy conditions are equal to Eqs. (77)–(78), given by

$$\Delta \ge 0, \quad (86)$$

and

$$\mu - p + 2p_t - \zeta\Theta + \Delta \ge 0. \quad (87)$$

6.4. Summary of the energy conditions

Summarizing the results, we rewrite the energy conditions. The energy conditions for a spherically symmetric fluid whose energy-momentum tensor is given by Eq. (3) are fulfilled if the following inequalities are satisfied:

$$(i) \quad |\mu + p - \zeta\Theta| - 2|q| \ge 0, \quad (88)$$

$$(ii) \quad \mu - p + 2p_t + \Delta - \zeta\Theta \ge 0, \quad (89)$$

and besides,

a) for the weak energy conditions

$$(iii) \quad \mu - p + \Delta + \zeta\Theta \ge 0, \quad (90)$$

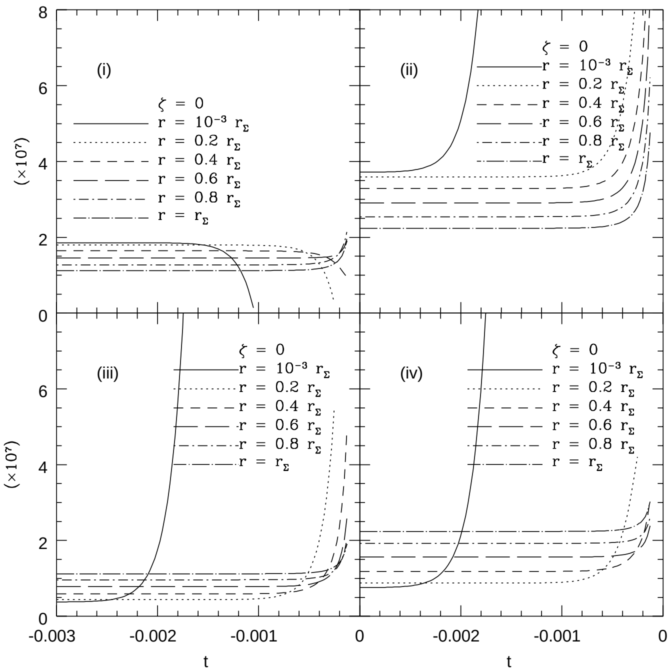

Fig. 7. The energy conditions (88)–(90), for the model without bulk viscosity, where $\zeta = 0$. The time is in units of seconds and all the others quantities are in units of s^{-2}

b) for the dominant energy conditions

$$(iv) \quad \mu - p + \zeta\Theta \ge 0, \quad (91)$$

$$(v) \quad \mu - p - 2p_t + \Delta + 3\zeta\Theta \ge 0, \quad (92)$$

c) for the strong energy conditions

$$(vi) \quad 2p_t + \Delta - 2\zeta\Theta \ge 0, \quad (93)$$

where $\Delta = \sqrt{(\mu + p - \zeta\Theta)^2 - 4q^2}$.

In order to verify the energy conditions, we have plotted the time evolution for all the conditions, for several radii and for two values of ζ (0 and 1000), as we can see in Figs. 7–9. For the sake of comparison with the model $\zeta \ne 0$, we have plotted all the conditions (88)–(93) for $\zeta = 0$.

From Figs. 7(i) and 8(i) we can conclude that only the inequality $[|\mu + p - \zeta\Theta| - 2|q| \ge 0]$ is not satisfied during all the collapse and for any radius. This inequality is not satisfied for the innermost radii ($r \le 0.2r_\Sigma$) and for the latest stages of the collapse. The condition (93) is not satisfied for $r < 0.2r_\Sigma$ [Figs. 9(vi) and 9(vi)] because the inequality (88) $[\Delta \ge 0]$ is not satisfied for these radii and for the latest stages of the collapse.

7. Physical results

As in Chan (1997, 1998, 2000), we have calculated several physical quantities, as the total energy entrapped inside the Σ surface, the total luminosity perceived by an observer at rest at infinity and the effective adiabatic index, and we have compared them to the respective non-viscous ones.

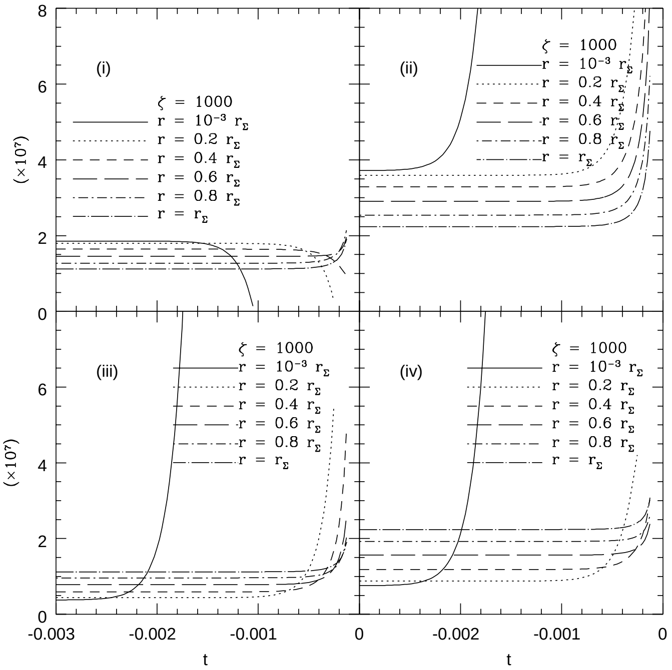

Fig. 8. The energy conditions (88)–(90), for the model with bulk viscosity, where $\zeta = 1000$. The time is in units of seconds and all the others quantities are in units of s^{-2}

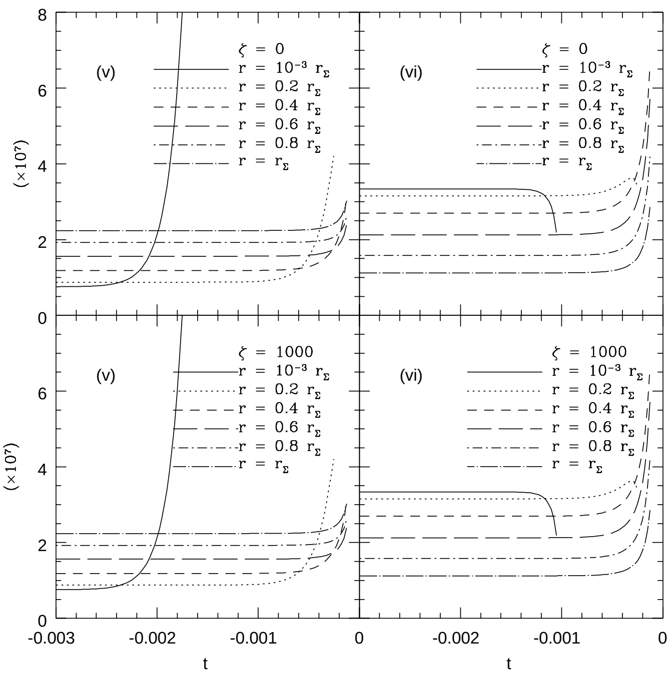

Fig. 9. The energy conditions (91)–(93), for the model with and without bulk viscosity, where $\zeta = 0$ and $\zeta = 1000$. The time is in units of seconds and all the others quantities are in units of s^{-2}

From Eq. (28) we can write using (37)–(39) that

$$m = \left[\frac{r^3 B_0^3}{2A_0^2} f \dot{f}^2 + \frac{r B_0}{2} f(1 - f^2) + m_0 f^3\right]_\Sigma , \qquad (94)$$

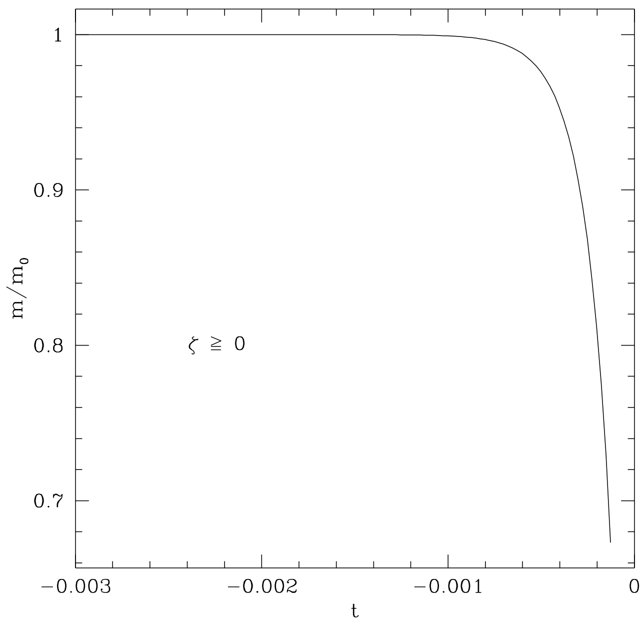

Fig. 10. Time behavior of the total energy entrapped inside the surface Σ for the models with and without bulk viscosity. The time, m and m_0 are in units of seconds

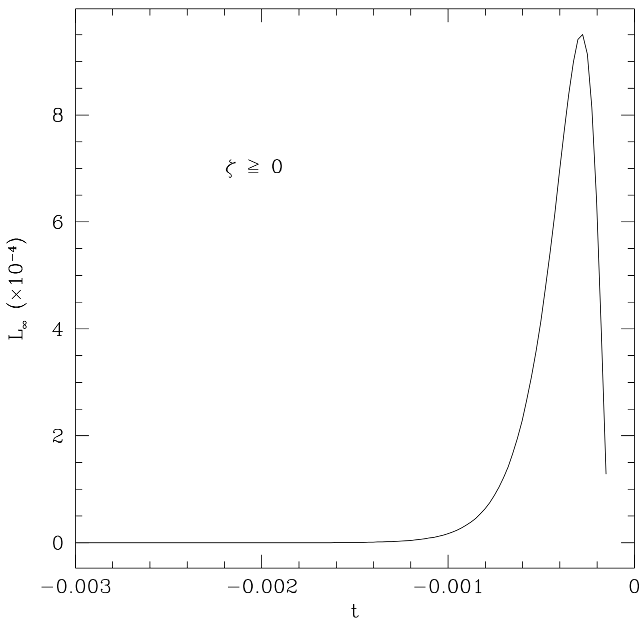

Fig. 11. Time behavior of the luminosity perceived by an observer at rest at infinity for the models with and without bulk viscosity. The time is in units of second and the luminosity is dimensionless

where

$$m_0 = -\left[r^2 B_0' + \frac{r^3 B_0'^2}{2B_0}\right]_\Sigma . \qquad (95)$$

We can observe from Fig. 10 that the mass inside Σ is equal for both models, with and without bulk viscosity. This means that they radiate the same amount of mass during the evolution.

Using Eqs. (34) and (37)–(39) we can write the luminosity of the star as

$$L_\infty = \frac{\kappa}{2}\left\{(p-\zeta\Theta)r^2B_0^2f^2 \times\left[\left(r\frac{B_0'}{B_0}+1\right)f+\left(\frac{rB_0}{A_0}\right)\dot{f}\right]^2\right\}_\Sigma . \quad (96)$$

This equation apparently depends on the viscosity but if we substitute Eqs. (40) and (42) into (96) the viscosity dependence vanishes remaining only the pressure without bulk viscosity. That is the reason we have presented in Fig. 11 a single plot for both models.

The effective adiabatic index can be calculated using Eqs. (41)–(42), (47) and (50)–(55). Thus, we can write that

$$\Gamma_{\rm eff} = \left[\frac{\partial(\ln p)}{\partial(\ln\mu)}\right]_{r={\rm const}} = \left(\frac{\dot p}{p}\right)\left(\frac{\mu}{\dot\mu}\right) =$$

$$\frac{c(r)f[3j(r)\dot{f}^2+bj(r)(1-f^2)]+\dot{f}\{c(r)[12b+aj(r)f^2]-12d(r)]\}}{c(r)[6\dot{f}^3+2af\dot{f}^2]+\dot{f}[2c(r)b(1-f^2)+4d(r)]}$$

$$\times\frac{12r^2d(r)f^2+h(r)[c(r)\dot{f}^2+d(r)(1-f^2)]}{72r^2e(r)f^2+h(r)\{c(r)j(r)f\dot{f}+3[bc(r)-d(r)](1-f^2)\}}, \quad (97)$$

where

$$c(r) = r^2m_0^2(1+r_\Sigma^2)^8, \quad (98)$$

$$d(r) = r_\Sigma^2 g^2(r), \quad (99)$$

$$e(r) = r_\Sigma^6(r_\Sigma^2-r^2)g(r), \quad (100)$$

$$h(r) = (1+r^2)^2, \quad (101)$$

and

$$j(r) = 3a+6\kappa\zeta A_0. \quad (102)$$

Comparing the figures for $\Gamma_{\rm eff}$ ($\zeta=0$ and $\zeta\neq 0$) we notice that the time evolution of the effective adiabatic indices are not very different graphically. This is the reason to plot the quantity $\delta\Gamma=\Gamma_{\rm eff}(\zeta=0)-\Gamma_{\rm eff}(\zeta\neq 0)$ instead of $\Gamma_{\rm eff}$ for the $\zeta\neq 0$ models. We can note in Fig. 12 ($\zeta=0$) that shortly before the peak of luminosity (see Fig. 11) there is a large discontinuity in $\Gamma_{\rm eff}$ due mainly to the behavior of the pressure. The effect of the viscosity is to increase much more these discontinuities, but diminishing the values of the indices.

8. Kretschmann scalar

The Kretschmann scalar is defined as

$$K = R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma\lambda}, \quad (103)$$

where $R_{\alpha\beta\gamma\lambda}$ are the components of the Riemann tensor. As suggested by Henry (2000), this scalar that characterizes the spacetime curvature allows us to better "visualize" the black hole.

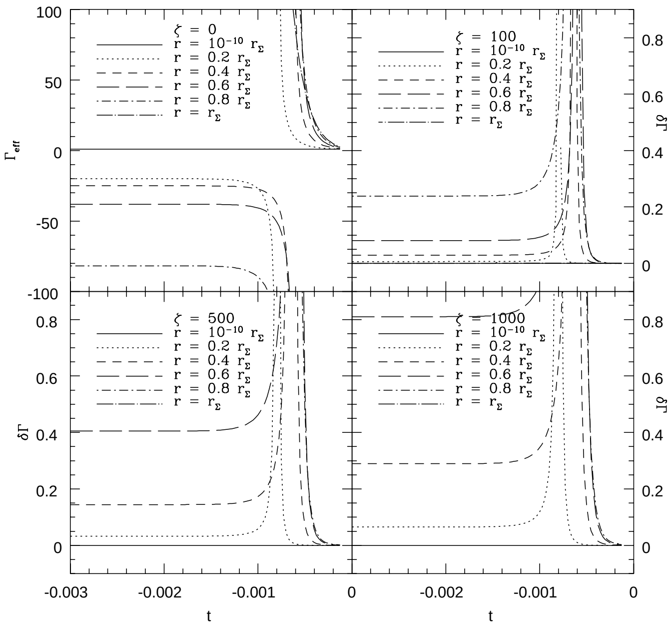

Fig. 12. Time behavior of the effective adiabatic index $\Gamma_{\rm eff}$ for four values of ζ. The quantity $\delta\Gamma$ is defined as $\Gamma_{\rm eff}(\zeta=0)-\Gamma_{\rm eff}(\zeta\neq 0)$. The time is in units of seconds, $\Gamma_{\rm eff}$ and $\delta\Gamma$ are dimensionless

Calculating the Kretschmann scalar using the REDUCE algebraic computing system, we get that

$$K = \frac{4}{A_0^4B_0^4r^4f^4} \times\left\{\left\{\left[3\left(\frac{A_0'}{A_0}\frac{B_0'}{B_0}\right)^2-2\left(\frac{A_0'}{A_0}\right)\left(\frac{A_0''}{A_0}\right)\left(\frac{B_0'}{B_0}\right)+\left(\frac{A_0''}{A_0}\right)^2 +3\left(\frac{B_0'}{B_0}\right)^4-4\left(\frac{B_0'}{B_0}\right)^2\left(\frac{B_0''}{B_0}\right)+2\left(\frac{B_0''}{B_0}\right)^2\right]r^4 +2r^2\left[\left(\frac{A_0'}{A_0}\right)^2+4\left(\frac{B_0'}{B_0}\right)^2\right]+4r^3\frac{B_0'}{B_0}\left[\left(\frac{A_0'}{A_0}\right)^2+\frac{B_0''}{B_0}\right] +4r\frac{B_0'}{B_0}+1\right\}A_0^4f^4-2r^2\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)^2A_0^4f^2+A_0^4 -2r^2A_0^2B_0^2\left\{\left[2r^2\left(\frac{A_0'}{A_0}\right)^2-4r^2\left(\frac{A_0'}{A_0}\frac{B_0'}{B_0}\right)+3r^2\left(\frac{B_0'}{B_0}\right)^2 -4r\frac{A_0'}{A_0}+6r\frac{B_0'}{B_0}+3\right]\dot{f}^2f^2+2r^2\frac{A_0'}{A_0}\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)\ddot{f}f^3-\dot{f}^2\right\} +r^4B_0^4\left(2f^2\ddot{f}^2+\dot{f}^4\right)\right\}. \quad (104)$$

We note in Eq. (104) that the most negative exponent term in f is $\frac{4}{r^4B_0^4f^4}$, thus, when f or/and r go to zero the Kretschmann scalar diverges.

Substituting Eqs. (47)–(53) into (104) and calculating numerically the Kretschmann scalar, we can get the plot shown in Fig. 13. As we can see from this figure, the Kretschmann scalar has no negative values, i.e., no negative curvature and diverges when t or/and r go to zero. Although it is not our case, this is in contrast to the

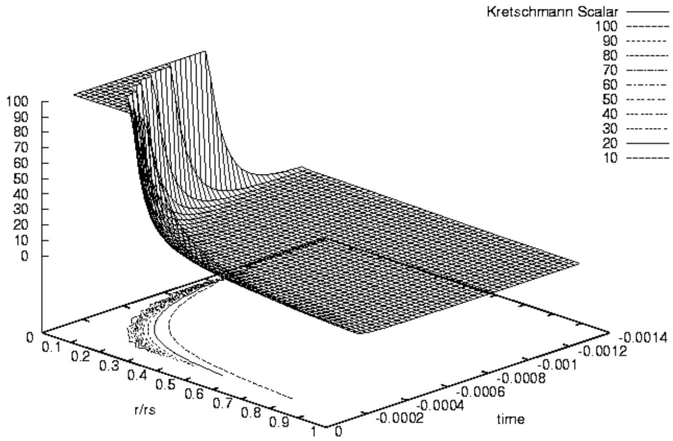

Fig. 13. Plot of the Kretschmann scalar as a function of the radius r (rs denotes r_Σ) and the time t, all in units of second. The curves on the base represent the isovalues of K. All the values of K are divided by a factor 10^{17} and the plot is truncated to values greater than 100 10^{17}

results of Henry (2000), where he found that rotating black holes possess a negative curvature.

9. Conclusions

The main conclusions of the present study are:

1 - We have generalized the result that the pressure has non zero value at the surface of the star unless the heat flux and bulk viscosity vanish;
2 - The pressure anisotropy increases with the bulk viscosity;
3 - The total radiated mass is equal for the non-viscous star and for the star with bulk viscosity;
4 - The star luminosity is the same for both collapsing models;
5 - The collapsing times with bulk viscosity and without bulk viscosity are the same;
6 - The bulk viscosity decreases the value of the effective adiabatic index;
7 - The Kretschmann scalar has no negative values.

Acknowledgements. The author acknowledges the financial support from the Brazilian CAPES (No. BEX 1097/99).

References

Bonnor, W. B., de Oliveira, A. K. G. & Santos, N. O. 1989, Phys. Rep., 181, 269
Cahill, M. E., & McVittie, G. C. 1970, J. Math. Phys., 11, 1382
Chan, R. 1993, Ap&SS, 206, 219
Chan, R. 1997, MNRAS, 288, 589
Chan, R. 1998, MNRAS, 299, 811
Chan, R. 2000, MNRAS, 316, 588
Chan, R., Herrera, L., & Santos, N. O. 1994, MNRAS, 267, 637
Cutler, C., & Lindblom, L. 1987, ApJ, 314, 234
de Oliveira, A. K. G., Santos, N. O., & Kolassis, C. A. 1985, MNRAS, 216, 1001
Henry, R. C. 2000, ApJ, 535, 350
Herrera, L., & Santos, N. O. 1997, Phys. Rep., 286, 53
Israel, W. 1966a, Nuovo Cimento, 44B, 1
Israel, W. 1966b, Nuovo Cimento, 48B, 463
Kolassis, C. A., Santos, N. O., & Tsoubelis, D. 1988, Class. Quantum Grav., 5, 1329
Martínez, J., & Pavón, D. 1994, MNRAS, 268, 654
Raychaudhuri, A. K., & Maiti, S. R. 1979, J. Math. Phys., 20, 245
Santos, N. O. 1985, MNRAS, 216, 403
Vaidya, P. C. 1953, Nature, 171, 260
Woosley, S. E., & Phillips, M. M. 1988, Science, 240, 750