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Ambipolar diffusion in self-gravitating filaments

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Abstract. One-dimensional similarity solutions for a collapsing, homogeneous, infinitely long cylinder that is subject to ambipolar diffusion are given. There is an analytical solution with infinite density being reached after 4.3 free-fall times. In that case the magnetic field strength B_z on the axis scales like $B_z \propto \rho^{2/3}$ with density ρ. The analytical solution proves "attractive". Even if the initial conditions depart slightly from those of the analytical solution, that solution is nevertheless approached through damped oscillations.

Key words. magnetohydrodynamics (MHD) – stars: formation – ISM: clouds – ISM: magnetic fields

1. Introduction

Observations of molecular clouds, the sites of present-day star formation, reveal dense clumps and slender filaments on many different scales; e.g. Falgarone et al. (1992, 2001), and for an overview see Mac Low & Klessen (2004). Large-scale, as well as small-scale, magnetic fields are probably important (Crutcher 1999), as they may play a vital role in the evolution of interstellar clouds and, therefore, in the onset of protostar formation. Besides their ability to extract angular momentum, they would even be able to support molecular gas well enough against gravitational collapse, were it not for ambipolar diffusion (Mestel & Spitzer 1956; Mouschovias 1978), the magnetic field slippage relative to the neutral matter in weakly ionized matter. The case of an initially magnetically subcritical cloud core being converted into a supercritical one as magnetic flux leaks out has been extensively considered by Mouschovias and collaborators (see Basu & Mouschovias 1995a,b, and references therein). Within the so-called standard theory such subcritical cloud cores are the sites for loss-mass star formation. For a differing view see Nakano (1998) and Mac Low & Klessen (2004). The *combined* effects of supersonic turbulence, strong magnetic fields, and ambipolar diffusion on cloud evolution are considered by Li & Nakamura (2004).

In view of the difficulties one envisages dealing with the gravitational collapse of magnetized spheroidal or disk-like (Barker & Mestel 1996; Basu & Ciolek 2004) proto-stellar cloud cores, it is perhaps of interest to see how the corresponding problem in a homogeneous infinitely long cylinder can be fully solved analytically, because the collapse proves always quasi-static. Numerically, the problem of ambipolar diffusion in self-gravitating infinite cylinders has been tackled at length in several papers by Mouschovias & Morton (1991, 1992a,b).

Moreover, in the case of a strong regular magnetic field the gravitational instability is considerably suppressed (Chandrasekhar & Fermi 1953; Nagasawa 1987; Fiege & Pudritz 2000b; Fiege 2003), so that ambipolar diffusion gains ample time to create high-density filaments as precursors to clump formation. Therefore, in order to understand the formation of globular filaments with a chain of knots within, the magnetic flux loss in filaments due to ambipolar diffusion needs to be taken into account.

Extending Ostriker's non-magnetic isothermal solution (Ostriker 1964), equilibrium configurations with a magnetic field in the direction of the major axis have been described by Stodólkiewicz (1963), while the case of helical fields is treated by Nakamura et al. (1993) and in a series of papers by Fiege & Pudritz (2000a,b,d,c).

The plasma and field drift radially through an infinitely long cylinder of neutral bulk matter is considered by Spitzer (1978). He finds a diffusion time of $t_\mathrm{D} = r/|u_\mathrm{D}| \approx 5.0 \times 10^{13} n_\mathrm{i}/n_\mathrm{H}$ years, where $u_\mathrm{D} = u - u_\mathrm{i}$ is the drift speed between neutrals and ions, r an initial radius, and n_i and n_H denote the density of ions and neutrals, respectively.

A brief outline of the paper is as follows. In Sect. 2 the equations describing ambipolar diffusion in the cylindrical case are set out. Analytical, as well as numerical, self-similar solutions of the full one-dimensional collapse problem for a homogeneous radially shrinking cylinder are given in Sect. 3. The astrophysical relevance is discussed in Sect. 4 followed by the conclusions (Sect. 5).

2. Equations

An infinitely long cylindrical filament is considered to extend into the z-direction and be pervaded by an aligned magnetic field B_z. A Lagrangian description is then adopted.

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All quantities depend only on radial mass coordinate $M = \int_0^r \rho\, r'\, \mathrm{d}r'$ and time t. M is the mass per unit length divided by 2π. Any z-dependence is neglected. For this one-dimensional time-dependent problem, the conservation, momentum, and induction equations in a single-fluid approximation read as follows:

$$\frac{\partial \rho}{\partial t} = -\rho^2 \frac{\partial}{\partial M}(r\, u)\,, \tag{1}$$

$$\frac{\partial u}{\partial t} = -\frac{r}{8\pi}\frac{\partial B_z^2}{\partial M} - r\frac{\partial p}{\partial M} - \frac{4\pi G}{r}\cdot M, \tag{2}$$

$$\frac{\partial r}{\partial t} = u, \tag{3}$$

$$\frac{\partial B_z}{\partial t} = r\rho\,(u - u_\mathrm{i})\frac{\partial B_z}{\partial M} - \rho B_z \frac{\partial}{\partial M}(r\, u_\mathrm{i})\,. \tag{4}$$

All quantities have their usual meaning, and u_i denotes the radial velocity of the ions.

The magnetic field is assumed to be frozen in the plasma of ions and electrons. The effect of charged grains, i.e. a stronger coupling of the magnetic field to the neutral matter, is neglected here (cf. Ciolek & Basu 2001). The magnetic flux remains conserved with respect to the plasma. The friction term in the momentum Eq. (2) has been already substituted by the Lorentz force. This is appropriate, as in the limit of low ionization, the mass density of the conducting fluid is negligible as compared to the mass density of the neutral bulk matter.

Equating Lorentz force and frictional drag, one gets the drift velocity

$$u_\mathrm{i} - u = -\frac{r}{8\pi\gamma\rho_\mathrm{i}}\,\frac{\partial B_z^2}{\partial M}, \tag{5}$$

where $\gamma = 3.5\times10^{13}\ \mathrm{cm^3\,g^{-1}\,s^{-1}}$ is the drag coefficient and ρ_i denotes the density of the ions.

The low ionization in molecular clouds is mainly due to cosmic rays. Here we use the canonical expression $\rho_\mathrm{i} = C\rho^{1/2}$, with $C = 3\times10^{-16}\ \mathrm{cm^{-3/2}\,g^{1/2}}$. The square-root dependency follows from the ionization-recombination equilibrium (Elmegreen 1979). All numerical values are from Shu (1992). In terms of γ and C Spitzer's diffusion time is $t_\mathrm{D} = \gamma C/(2\pi G\sqrt{\rho})$. The scaling $\rho_\mathrm{i} \propto \rho^{1/2}$ is a reasonable approximation, but it is not always true (cf. Ciolek & Mouschovias 1996; Caselli et al. 2002).

Using appropriate units for time, magnetic field strength, and pressure, viz. $t_0 = (4\pi G\rho_0)^{-1/2}, B_0 = 4\pi\sqrt{2G}\,\rho_0 r_0, p_0 = 4\pi G\rho_0^2 r_0^2$, one arrives at normalized equations:

$$\frac{\partial \rho}{\partial t} = -\rho^2 \frac{\partial}{\partial m}(s\, u)\,, \tag{6}$$

$$\frac{\partial u}{\partial t} = -s\frac{\partial B_z^2}{\partial m} - s\frac{\partial p}{\partial m} - \frac{m}{s}, \tag{7}$$

$$\frac{\partial s}{\partial t} = u, \tag{8}$$

$$\frac{\partial B_z}{\partial t} = s\rho\,(u - u_\mathrm{i})\frac{\partial B_z}{\partial m} - \rho B_z \frac{\partial}{\partial m}(s\, u_\mathrm{i})\,,\ \text{and} \tag{9}$$

$$u_\mathrm{i} - u = -\alpha_\mathrm{AD}\,\frac{2\,s}{3\rho^{1/2}}\,\frac{\partial B_z^2}{\partial m}, \tag{10}$$

where $m = M/(\rho_0 r_0^2)$, $s = r/r_0$ and velocities are in units of r_0/t_0. The time-scale $t_0 = t_\mathrm{ff}/\sqrt{\pi}$ is essentially the free-fall time t_ff of a homogeneous cylinder. The coefficient $\alpha_\mathrm{AD} = 3\sqrt{\pi G}/(\gamma C) = 0.131$ measures the strength of ambipolar diffusion. In the notation of Mouschovias & Morton (1991) α_AD is related to their ν_ff, the ratio of free-fall and neutral-ion collision time scales divided by $\sqrt{\pi}$, according to $\alpha_\mathrm{AD} = 2.1/\nu_\mathrm{ff}$. There is no free parameter in the friction Eq. (10), because the ratio of diffusion to free-fall time is fixed in the cylindrical case: $t_\mathrm{D}/t_\mathrm{ff} = \gamma C/(\pi\sqrt{G}) \simeq 12.9$. Both diffusion time and free-fall time scale as $\rho^{-1/2}$. That the ambipolar diffusion time is on the order of the free-fall time is not at all restricted to our special model, an infinite cylinder. It also does not depend on geometry but results solely from the ionization equilibrium, i.e. $\rho_\mathrm{i} \propto \rho^{1/2}$, as Mouschovias & Morton (1991) have shown. At the axis ($s = 0$) $u = u_\mathrm{i} = 0$, $\partial p/\partial m = 0$ and $\partial B_z/\partial m = 0$ holds.

3. Similarity solutions

3.1. A set of two ordinary differential equations

The set of Eqs. (6)–(10) is solved by the ansatz:

$$\rho(m, t) = m^{-\alpha}\cdot f(t),\quad \alpha > -1. \tag{11}$$

From the equation of continuity (6) then follows:

$$s^2 = \frac{2\,m^{1+\alpha}}{(1+\alpha)f(t)}. \tag{12}$$

Hence, the bulk velocity depends linearly on s:

$$u = \frac{\partial s}{\partial t} = -\frac{s}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\log f. \tag{13}$$

The ionic velocity can be parameterized by a time-dependent $\delta(t)$:

$$u_\mathrm{i} = (1-\delta)\cdot u. \tag{14}$$

After some cumbersome algebraic manipulations, one obtains two coupled ordinary differential equations that describe the evolution of density $f(t)$ and of relative drift velocity $\delta(t)$. It turns out that the momentum Eq. (7) requires $\alpha = 0$; i.e. the density is distributed homogeneously. For any barotropic equation of state, $p = p(\rho)$, the pressure gradient vanishes then, too.

$$\frac{\dot\delta}{\delta} = -\frac{\ddot f}{\dot f} + \frac{\dot f}{f}\cdot(5/2 - 3\,\delta)\,, \tag{15}$$

$$\frac{\ddot f}{f} - \frac{3}{2}\left(\frac{\dot f}{f}\right)^2 = -\frac{3\,\delta}{2\,\alpha_\mathrm{AD}}\cdot\frac{\dot f}{\sqrt{f}} + f. \tag{16}$$

A dot means differentiation with respect to time t. For details the reader is referred to the appendix.

3.2. An analytical solution

Provided the magnetic field gradient always balances the self-gravity term, the Lagrangian velocity u is conserved. The contraction remains *quasi-static*. This happens if $\delta = 1/3$:

$$f(t) = (1 - \alpha_\mathrm{AD}\cdot t)^{-2}\,. \tag{17}$$

The velocity fields evolve according to

$$u(s,t) = -\frac{\alpha_{\rm AD}}{(1-\alpha_{\rm AD}\,t)} \cdot s \quad \text{and} \quad u_{\rm i}(s,t) = \frac{2}{3} \cdot u(s,t). \tag{18}$$

Hence, the collapse time $t_{\rm ff}/(\alpha_{\rm AD}\sqrt{\pi})$ is 4.3 free-fall times or a third of Spitzer's diffusion time $t_{\rm D}/3$. It should be noted that the initial state $t = 0$ is already collapsing.

The magnetic field strength depends on radius in an ellipsoidal manner. Its radial extension is initially limited to $m \leq 2 \cdot B_z(0,0)^2$ or $s \leq 2 \cdot B_z(0,0)$, respectively, where $B_z(0,0)$ is an arbitrary initial field strength on the axis of the filament. The field within the constant-density filament is given by

$$B_z(m,t) = \frac{B_z(0,0)}{(1-\alpha_{\rm AD}\,t)^{4/3}} \cdot \sqrt{1 - \frac{m}{2}\frac{(1-\alpha_{\rm AD}\,t)^{2/3}}{B_z(0,0)^2}}. \tag{19}$$

Hence, the magnetic field strength on the axis scales with density as

$$B_z(0,t) \propto \rho^{2/3}. \tag{20}$$

Switching from m to r and to real time one gets

$$B_z(r,t) = \frac{B_z(0,0)}{(1-\alpha_{\rm AD}\,t/t_0)^{4/3}} \cdot \sqrt{1 - \frac{(r/r_{\rm max})^2}{(1-\alpha_{\rm AD}t/t_0)^{4/3}}}, \tag{21}$$

where $r_{\rm max}$ denotes the maximal possible initial radius $r_{\rm max} = B_z(0,0)/(2\pi\sqrt{2G}\rho_0)$ or, in convenient units, $r_{\rm max} = 0.38\,{\rm pc}\,(B_z(0,0)/10\,\mu{\rm G})(10^3\,{\rm cm}^{-3}/n)$, with n being the number density of the neutrals. (The neutrals' mean mass has been assumed 2.3 times the mass of a hydrogen atom.) In terms of the Alfvén speed of the neutral fluid, $u_{\rm A} = B_z/\sqrt{4\pi\rho_0}$, and the free-fall time, $t_{\rm ff} = 1/\sqrt{4G\rho_0}$, $r_{\rm max}$ is on the order of Mouschovias' (1991) magnetic Jeans length: $r_{\rm max} = \sqrt{2/\pi}\,u_{\rm A}\,t_{\rm ff}$.

It is worth noting that, as time goes on, the radial extension of the plasma cylinder with its frozen-in magnetic field necessarily exceeds the radius of the collapsing neutral gas cylinder. Plasma which left the constant density filament feels a reduced drag force and will be, in any case, left behind the collapsing neutral matter. Regardless of what happens with the plasma in the exterior, it cannot further influence the filament. That is why the solution should aptly describe the fast contraction of the constant density core of a real molecular cloud filament.

3.3. Numerical self-similar solutions

If the initial velocity fields deviate from Eq. (18), the evolution can nevertheless be followed by integrating Eqs. (15) and (16) numerically, subject to $f_0 = 1$. The solution depends solely on the initial values of δ_0 and $\dot{f}_0$.

In order to find out how far the exact solution (17) proves "attractive", integrations with starting values $(\delta_0, \dot{f}_0)$ not so far away from the similarity case $(1/3, 2\,\alpha_{\rm AD})$ were performed. To get rid of $\dot{f}_0$, only δ_0 was varied. $\dot{f}_0(>0)$ was chosen in such a way that $\dot{\delta}_0$ vanishes. For $\delta_0 < 1/3$ this procedure is unequivocal. The resulting $\dot{f}_0$ is single-valued. Forcing $\dot{\delta}_0 = 0$ prevents the starting time from being a privileged one in a certain sense. Two examples are given in Fig. 1, where δ has been initially set

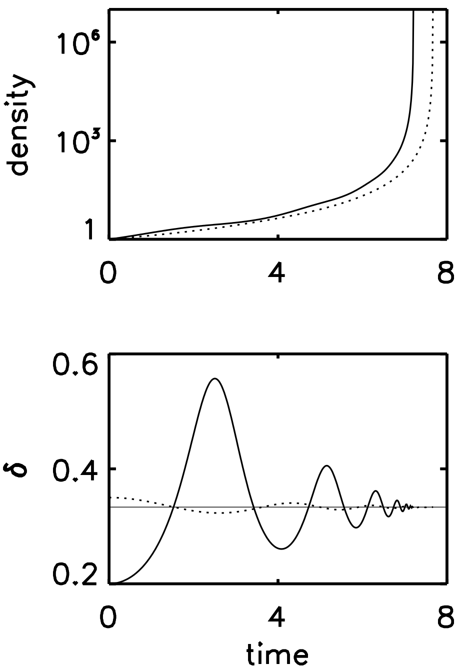

Fig. 1. Density rise $f(t)$ and evolution of the δ-parameter for two δ_0-values: 0.2 (solid line) and 0.35 (dotted line). Both curves approach $\delta = 1/3$ (horizontal line) in the upshot. The exact solution (17) itself reaches infinity at $t_\infty = 7.64$ (or 4.31 free-fall times); i.e. it lies in between the two density curves

to $\delta_0 = 0.2$ and 0.35. The fast contracting filaments are obviously stable against radial oscillations.

Relaxation via damped oscillations toward the exact solution (17) proves typical as long as the value of δ_0 does not depart too much from 1/3. For $\delta_0 \lesssim 0.12$, no satisfying solution was found. The collapse comes to a premature stand-still; i.e. the density approaches a finite value at the expense of a faster and faster outward flow of ions, ultimately violating the assumption of slow drift velocities. The drag force depends linearly on the drift velocity only as long as $u_{\rm d} \lesssim 10\,{\rm km\,s}^{-1}$ (see Mouschovias & Paleologou 1981; Shu 1992).

It may be noted here that the same behaviour, i.e. a ceasing collapse with a soaring-up δ, has been found in the case of a density-independent diffusion time $t_{\rm D}$, too, i.e. ambipolar diffusion with fixed ionizational fraction. The self-similar route to infinity discussed here, i.e. the case of quasi-static contraction, relies solely on putting $\rho_{\rm i} \propto \rho^{1/2}$!

Starting from rest, i.e. $\dot{f}_0 = 0$, is impossible (cf. Eq. (A.7)), but one can try to commence with $\dot{f}_0$ as low as numerically possible. The neutrals are then practically at rest and gravity is balanced by (in terms of δ) an initially outward flowing plasma. In that case of vanishing acceleration $\delta_0 = 2\,\alpha_{\rm AD}/(3\dot{f}_0)$ is, of course, very high. Nevertheless, after 0.45 free-fall times the ions are collapsing, too. For $\dot{f}_0$ approaching 0 the numerically found collapse times are 4.55 free-fall times, which is only marginally longer than the collapse time for the analytical self-similar solution, viz. 4.31 $t_{\rm ff}$ (all numbers are basing on $\alpha_{\rm AD} = 0.131$).

4. Astrophysical application

Our self-similar collapse solutions are limited to the case of a *homogeneous* cylinder, a requirement, which in view of the observed steeply declining density profiles, seems to be far from reality. Yet there must be a constant density core. The Ostriker (1964) solution for an isothermal self-gravitating filament exemplifies exactly that – a core-envelope structure. It is the contraction of such a homogeneous core, embedded in a high-pressure environment, to which our solution might apply. In order to see whether the self-similar solution reflects some properties of more realistic core-envelope configurations, one should compare it with the extensive numerical simulations done by Mouschovias & Morton (1992a,b). Especially interesting is their "cold" Model "10", because of the vanishing thermal pressure. Without any pressure support, even this model evolves quasi-statically; i.e. the acceleration vanishes after an initial adjustment phase. As noted by the authors the evolution of the cores generally turns out rather insensitive to the initial conditions and becomes more so as time goes on. By least-squares fitting the following representation of the density evolution of their model "10" is found: $f = (1 - t/(4.13 \cdot t_{\rm ff}))^{-2.047}$. The mean deviation of the fitted (logarithms of) densities from the values communicated in the paper is 0.026 dex (or 6.3%). Hence, the central density evolves almost exactly according to Eq. (17)! If one starts from an already ten-fold density increase, to exclude the initial adjustment phase, one gets $f = (1 - t/(4.07 \cdot t_{\rm ff}))^{-2.086}$. Remember that $f = (1 - H_0 \cdot t)^{-2}$ happens if at $t = 0$ the infall velocity obeys a "Hubble law", $u = H_0 \cdot r$, with Lagrangian velocity u being conserved. The remaining collapse time in this case of vanishing acceleration is H_0^{-1}. In that simple case the density time-scale; i.e. $\tau_{\rho,\rm c}$ in Fig. 9c of Mouschovias & Morton (1992a), goes with $\rho_{\rm c}^{-1/2}$. This is indeed the case. In their graph the slope is −0.48.

What about the collapse time itself? First, one has to adjust the microphysics somewhat. In the standard model of Mouschovias & Morton (1991) the coupling is weaker, so that the effect of ambipolar diffusion, as expressed by $\alpha_{\rm AD}$, is stronger: $\alpha_{\rm AD} = 0.253$ instead of 0.131. Therefore, one expects shorter collapse time by a factor of two, i.e. 2.23 $\tau_{\rm ff}$. But this is not the case. The numerically found collapse time is instead compatible with our $\alpha_{\rm AD} = 0.131$ rather than with their $\alpha_{\rm AD} = 0.253$. The reason for this discrepancy is perhaps that their models do not have a spatially uniform density as assumed here. Another point of concern is the $B_{\rm c} - \rho_{\rm c}$ relation. Numerically $B_{\rm c} \propto \rho_{\rm c}^{0.4}$ has been found, whereas analytically the magnetic field strength on the axis scales according to $B_{\rm c} \propto \rho_{\rm c}^{2/3}$ (cf. Eq. (20)). The scaling found by the observers, $B_{\rm c} \propto \rho_{\rm c}^{1/2}$, lies in between (Crutcher 1999).

A crucial point concerns the stability issue of whether or not ambipolar diffusion is fast enough to compete with gravitational knot formation. Despite the fact that a cylindrical filament is basically gravitationally unstable with respect to long-wavelength perturbations, our case of a radially *fast* contracting cylinder seems to be feasible for the following reason. As Nagasawa (1987) has shown, external pressure, combined with a strong poloidal magnetic field, stabilizes an otherwise fragile isothermal cylinder considerably. If the cut-off radius is smaller than the core radius of Ostriker's solution, the rise-time of the fastest growing mode grows exponentially with B_z^2. The stability behaviour then becomes similar to that of the incompressible cylinder already studied by Chandrasekhar & Fermi (1953). More general cases have been considered by Nakamura et al. (1993), Fiege & Pudritz (2000b), and Fiege (2003). There seems to be ample time for ambipolar diffusion to allow for radial contraction before a filament breaks up into knots. In the case of a twisted field, a complication ignored here, there would of course be the possibility of a kink instability, too.

5. Conclusions

The magnetic support of a filament's core, consisting of a weakly ionized molecular gas, will prolong its life-time by a factor of only 4–5 as compared with the free-fall time. This confirms the general result that the collapse retardation factor must be of order unity, whether the collapse is quasi-statical or not and independent of geometry, a notion which follows from first principles (Mouschovias 1987; Mouschovias & Ciolek 1999; Ciolek & Basu 2001). It is this rapidity of the ambipolar diffusion which makes the notion of a long filament contracting almost purely in radial direction feasible. Gravitational fragmentation, i.e. the formation of clumps which might eventually evolve into star-forming spheroidal cores, probably needs a longer time span. Even if the initial conditions are incompatible with the exact self-similar solution (17), the latter may be approached via strong relaxational oscillations.

In contrast to ambipolar diffusion in a spheroidal protostellar cloud, there is no transition from a quasi-static contraction to a fast proceeding collapse. A strong leakage of magnetic field ($\delta \simeq 1/3$!) is necessary to prevent the magnetic field in a contracting cylinder to become energetically too important, for the gravitational energy per length of a filament does not depend on its radius (see McCrea 1957).

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Appendix A: Similarity solutions

A.1. Without momentum equation

In order to solve the Set (6)–(10) we look for a similarity solution using the ansatz:

$$\rho(m,t) = m^{-\alpha} \cdot f(t), \quad \alpha > -1. \tag{A.1}$$

Since no loss of generality is implied by supposing $\rho(1,0) = 1$ (i.e. that the density at $m = 1$ be initially the unit density ρ_0), we shall write $f(0) = 1$. From the equation of continuity (6) follows:

$$s^2 = \frac{2\,m^{1+\alpha}}{(1+\alpha)f(t)}. \tag{A.2}$$

Hence, the bulk velocity depends linearly on s:

$$u = \frac{\partial s}{\partial t} = -\frac{s}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \log f. \tag{A.3}$$

The ionic velocity be parameterized by a time-dependent $\delta(t)$:

$$u_{\rm i} = (1-\delta)\cdot u. \tag{A.4}$$

In order to fulfill the friction Eq. (10) the parameterization (A.4) requires the magnetic field to take the following form

$$B_z^2 = \bar{B}_z^2 - \frac{3\delta}{4\alpha_{\rm AD}}\sqrt{f}\cdot\frac{\rm d}{{\rm d}t}\log f\cdot\frac{m^{1-\alpha/2}}{1-\alpha/2}, \tag{A.5}$$

with $\bar{B}_z$ only depending on time t.

The time derivative of the magnetic field (A.5) must satisfy the induction Eq. (9). One finds that the homogeneous part of the magnetic field $\bar{B}_z(t)$ evolves according to

$$\frac{\rm d}{{\rm d}t}\log\bar{B}_z = (1-\delta)\cdot\frac{\rm d}{{\rm d}t}\log f. \tag{A.6}$$

The evolution of the relative drift velocity $\delta(t)$ is given by

$$\frac{\dot\delta}{\delta} = -\frac{\ddot f}{\dot f} + \frac{\dot f}{f}\cdot(1-q), \tag{A.7}$$

where $q(t) = 3\,\delta(t)\cdot(1+\alpha/2)/(1+\alpha) - 3/2$ and a dot means differentiation with respect to time t. Hence, from a *given* density evolution $f(t)$, the evolution of the relative drift velocity $\delta(t) = (u-u_{\rm i})/u$ can be computed. This freedom, of course, is broken, if the momentum Eq. (7) is considered, too. But, before performing that last step, let us ponder Eq. (A.7) somewhat. As long as q (and hence δ) is bounded, Eq. (A.7) can be formally integrated by means of the mean value theorem of integral calculus

$$\frac{\delta}{\delta_0}\cdot\frac{\dot f}{\dot f_0} = f^{1-q^*(t)}, \tag{A.8}$$

where a subscript "0" means an initial value (i.e. at $t = 0$), and q^* is an unknown but restricted mean value. For q = const. (A.8) can be further integrated:

$$\begin{aligned} f(t) &= e^{t/t_{\rm ch}}, & q &= 0,\\ f(t) &= (1+q\cdot t/t_{\rm ch})^{1/q}, & &\text{else}. \end{aligned} \tag{A.9}$$

The time-scale $t_{\rm ch} = 1/\dot f_0$ is set by the initial velocity field: $u|_{t=0} = -s/(2\,t_{\rm ch})$. For $q < 0$ the collapse is finished at a finite time $t_\infty = t_{\rm ch}/(-q)$.

A.2. Integration of momentum equation

The momentum Eq. (7) restricts possible solutions with a power-law dependence of density on mass (A.1) to the homogeneous case: $\alpha = 0$. For any barotropic equation of state $p = p(\rho)$, the pressure gradient then vanishes, too. With acceleration $\ddot s = -s\cdot(\ddot f/f - 3(\dot f/f)^2/2)/2$, the momentum balance (7) then simplifies to

$$\frac{\ddot f}{f} - \frac{3}{2}\left(\frac{\dot f}{f}\right)^2 = -\frac{3\,\delta}{2\,\alpha_{\rm AD}}\cdot\frac{\dot f}{\sqrt{f}} + f. \tag{A.10}$$

Inserting the general solution (A.8) into (A.10) leads to a lengthy expression containing terms differing in the power of f: $A(\dot\delta,\dot q^*)\,f^{-q^*} + B(\delta,q^*,\dot f)\,f^{-2q^*} + C\,f^{1/2-q^*} + f = 0$. A vanishes for $\dot\delta = 0$. Thus, there is a single route to infinity, namely fixing $q^* = -1/2$.

An integral of Eq. (A.10) is

$$\dot f\,f^{-3/2} + \frac{3}{2\,\alpha_{\rm AD}}\int\delta\,\frac{{\rm d}f}{f} - \dot f_0 = \int\sqrt{f}\,{\rm d}t, \tag{A.11}$$

where $\dot f_0$ stands for $\dot f(0)$. Let us now consider the ratio R between the magnetic pressure term and the gravitational attraction

$$R = \frac{3\,\delta}{2\,\alpha_{\rm AD}}\cdot\frac{\dot f}{f^{3/2}}. \tag{A.12}$$

Its time derivative is, together with (A.7)

$$\dot R = \frac{3\,(1-3\,\delta)\,\delta}{2\,\alpha_{\rm AD}}\cdot\frac{\dot f^2}{f^{5/2}}. \tag{A.13}$$

As long as $0 < \delta < 1/3$ the magnetic pressure terms gains always ground as compared with the gravitational attraction. The contrary is true if $\delta > 1/3$. Hence, $\delta = 1/3$ acts like a watershed. It separates the region of growing magnetic influence, i.e. $0 < \delta < 1/3$, from the regime where gravity will eventually overtake any magnetic deceleration. The analytical solution (17) is the one where $R = 1$ always.

A further interesting property of $f(t)$ is its homogeneity in time. Given a solution $f_1(t)$ and $\delta_1(t)$, $f_2(t) = \xi^2\,f_1(\xi t)$ and $\delta_2(t) = \delta_1(\xi t)$ solves Eqs. (A.7) and (A.10), too. A density increase shortens the time-scale simply as given by the free-fall time. Restrictions result from combining Eq. (A.7) with Eq. (A.10). After some algebraic manipulations one gets

$$\left(\frac{\dot f}{f}\right)^2 + \frac{\dot\delta/\delta - 3\delta\sqrt{f}/(2\alpha_{\rm AD})}{3\delta-1}\cdot\frac{\dot f}{f} + \frac{f}{3\delta-1} = 0. \tag{A.14}$$

As a result, for $\delta > 1/3$ a solution is possible only if $(\dot\delta/\delta - 3\delta\sqrt{f}/(2\alpha_{\rm AD}))^2 > 4\cdot f\cdot(3\delta-1)$. This restricts the possible combinations of δ and $\dot\delta$ from the outset.

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