

Cosmological effects in the local static frame

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Received 18 October 2004 / Accepted 13 December 2004

Abstract. What is the influence of cosmology (the expansion law and its acceleration, the cosmological constant...) on the dynamics and optics of a local system like the solar system, a galaxy, a cluster or a supercluster? The answer requires the solution of the Einstein equation with the local source, which tends towards the cosmological model at large distances. There is, in general, no analytic expression for the corresponding metric, but we calculate here an expansion in a small parameter, which allows us to answer this question. First, we derive a static expression for the pure cosmological (Friedmann-Lemaître) metric, whose validity, although local, extends in a very large region around the observer. This expression appears as the metric of an osculating de Sitter model.

We then propose an expansion of the cosmological metric with a local source, which is valid in the very wide neighborhood of the local system. This allows us to calculate exactly the (tiny) influence of cosmology on the dynamics of the solar system: it results that, contrary to some claims, cosmological effects fail to account for the unexplained acceleration of the Pioneer probe by several orders of magnitudes. Our expression provides estimations of the cosmological influence in the calculations of rotation or dispersion velocity curves in galaxies, clusters, and any type of cosmic structure, necessary for precise evaluation of dark matter and/or cosmic flows. The same metric can also be used to estimate the influence of cosmology on gravitational optics in the vicinity of such systems.

Key words. gravitation – relativity – galaxies: kinematics and dynamics – cosmology: theory

1. Introduction

Motivation

Dynamical studies in the universe are usually separated into cosmological problems and local problems. For the first ones, the Friedmann-Lemaître (FL) metric g_{FL} is most often used, which is an exact solution of the the Einstein equations (that we consider in this paper as including the cosmological constant). For *local* problems, and when the gravitational field is weak (see below), one applies a Newtonian metric g_{Nw} , which is only an approximate solution of the Einstein equations for weak (“Newtonian”) fields.

It is clear that the validity of the Newtonian metric does not extend to cosmological distances from the source, even for weak fields; for instance, it would not account for the cosmological redshifts. It is a requirement of general relativity to reduce to Newtonian physics for *very local* (in a sense to be specified below) problems. However, the question of intermediate regimes remains open. The first goal of this paper is to specify

the meaning of “intermediate”, and to provide an approximation (since no exact solution exists) for it. This aims to answer the question: “what is the influence of cosmology on a local system?”.

Although cosmological effects are negligible in the solar system (this will be shown quantitatively below), they may influence the dynamics in galaxies, in the Local group or around clusters and superclusters. We intend here to evaluate exactly the influence of cosmology (the Hubble constant, the cosmological constant) inside such systems. For the Solar System for instance, it has been asked whether the unexplained acceleration of the Pioneer probe (Anderson et al. 1998; Turyshev et al. 2004) could be due to some cosmological effect. We will give a clear and quantitative (negative) answer by showing that such an effect remains tiny. This study evaluates the influence of cosmology on the dynamics in a galaxy, or in its neighborhood: for instance, what is the (weak) contribution of cosmology to the rotation curves? The cosmological influence increases with the size of the system. Thus, we may infer that it must be taken into account (at some order that we will define) for dynamical

studies inside or around galaxy clusters and superclusters, as well for the study of gravitational lensing involving these systems.

Small parameters

To evaluate the “local” character with respect to cosmology, we will define a natural parameter $\epsilon = r H_0$, where r approximates the size of the system (the distance to the gravitational source), and H_0 is the Hubble constant, whose inverse provides a cosmological distance scale. Fortunately, the other cosmological distance $\Lambda^{-1/2}$ is of the same order (the spatial curvature radius is probably even greater), so that this unique approximation will be valid for all cosmological effects. The Newtonian metric is a zero order term in ϵ . This paper explores the following terms, and gives their exact formulation up to second order, which is largely sufficient for all studies.

We emphasize that this development is distinct from the weak field approximation. In the latter, the small parameter may be taken as the potential ϕ_{Nw} (we chose units such that $c = 1$). The pure (FL) cosmological metric is at zero order. The Newtonian metric is at first order, the next order being the post-Newtonian one, that we do not consider in this paper (although a generalization to post-Newtonian order is straightforward). Thus our results will not apply in the neighborhood of compact objects like neutron stars or black holes. In the solar system, they will be valid only sufficiently far from the Sun. They apply without restrictions to galaxies, clusters or superclusters.

The form of the metric that we provide is perfectly adapted to analyse and interpret astrophysical results of cosmological relevance in these systems, and also to make the link with the usual concepts of laboratory physics.

Local developments

We consider a system with spherical symmetry, isolated in the Universe, to approximate the Sun, a galaxy or a cluster. An exact solution of the Einstein equations is given by the Schwarzschild metric g_{Schw} . Its first order approximation in ϕ_{Nw} gives g_{Nw} . It is clear that none of them takes into account any cosmological effect, like for instance the cosmological redshifts: g_{Schw} is the solution (of the Einstein equation) for an isolated source in a *static* universe without curvature, not in the expanding Universe. In fact, the solution has as a limiting condition a flat (Minkowski) space-time at infinity.

We search for the solution g of the same equation, with the limiting condition that the metric identifies to g_{FL} at infinity: g_{Schw} is its approximation at zero order in the parameter ϵ . Except in special cases, g cannot be obtained exactly. Our main result is to provide a static approximation G of g valid in a specified range.

A metric is independent of the choice of coordinates (a map). It is however conveniently expressed in a given map. One difficulty of the problem is to find a convenient map to express g or G . For a *global* analysis, the map where g_{FL} takes the usual Robertson-Walker form appears clearly the most convenient. This is not so for local studies. Thus, to facilitate local studies, we must first chose a map which provides a static form for the pure cosmological metric g_{FL} . Static solutions of the

Table 1. The different metrics at different orders with respect to the small parameters ϕ_{Nw} and ϵ .

Order	ϵ^0	ϵ^1	$\epsilon^2 \dots$
zero:	$(\phi_{\text{Nw}})^0$	$g_{\text{FL},0} = \eta_{\text{Mink}}$	$g_{\text{FL},1}$ $g_{\text{FL},2} \dots$ g_{FL}
Newtonian:	$(\phi_{\text{Nw}})^1$	g_{Nw}	G
exact:		g_{Schw}	g

Einstein equations are known and classified (Stephani 2003) but none identifies with g_{FL} , except in the particular case of de Sitter space-time. Thus, we are led to find a map which provides approximately (in a well defined sense) a static form of g_{FL} . We obtain this form G_{FL} in Sect. 2.2. It is exact at order ϵ^2 , which is largely sufficient for most calculations: even if we know the exact cosmological solution g_{FL} , we rewrite it in its approximate form G_{FL} to prepare the following.

This allows us to calculate (in Sect. 3.1) with the same approximation (at order ϵ^2), a static form G of g , the solution of the Einstein equations with the central source, and with the limiting FL conditions. We resume in the table the constraints which apply to the metric G :

- it is static;
- $G \approx g_{\text{Nw}}$ at order ϵ^0 ;
- $G \approx G_{\text{FL}}$ at zero order in ϕ_{Nw} (i.e., when the source is off).

Thus, G is the correct solution to explore the dynamics of the system beyond the near environment of the source.

The geometrical point of view

In plane geometry, the local study of a curve involves, at first order, its tangent and, at second order, its osculating circle. The situation is analogous to the space-time manifold (\mathcal{U}, g) . Its first order approximation, the (Minkowskian) tangent space, completely neglects any curvature (gravitational) effects: the local deformations due to local sources, as well as the imprint of the cosmic curvature (the accelerated cosmic expansion). Note that g_{Schw} takes only the local one and g_{FL} the cosmological one. It is of special importance to retain both, when one is interested in structures with a large extension (e.g., clusters or superclusters), where expansion affects the dynamics. The approximate metric G obtained below exactly provides the second order approximation of (\mathcal{U}, g) which takes all these effects into account. We claim that this is the best second order approximation for space-time, the analog of the osculating circle to a curve. As we will see, it appears to be of constant curvature and, thus, may be called the “osculating de Sitter” space-time.

Definitions

We recall some definitions.

An *inertial* frame is defined so that the time coordinate coincides with the proper time of a free-falling observer, implying

$$g_{00} = 1, \quad g_{0i} = 0. \quad (1)$$

A locally *inertial* system of coordinates around an event E is an inertial one such that the metric is the Minkowski metric

at E . (Weinberg 1972, p. 127) underlines the usefulness of this kind of frame to understand the stress-energy tensor: “note that p and ρ are always defined as the pressure and energy density measured by an observer in a locally inertial frame that happens to be moving with the fluid at the instant of measurement, and are therefore scalars”.

The principle of equivalence guarantees the existence of a local *inertial* form of any metric.

In a *static* system of coordinates, all coefficients of the metric are independent of the time coordinate.

2. The static expression of the cosmological metric

This section considers the purely cosmological case: we describe the Universe by a Friedmann-Lemaître model $(\mathcal{U}, g_{\text{FL}})$, i.e., spatially homogeneous and isotropic. Most often, g_{FL} is expressed in its Robertson-Walker form

$$g_{\text{FL}} = ds_{\text{FL}}^2 = d\tau^2 - R^2(\tau) \left[dx^2 + f_k^2(x) d\omega^2 \right], \quad (2)$$

where $f_k(x) = x, \sin(x), \sinh(x)$ according to the sign $k = 0, 1$ or -1 of the spatial curvature, and where $d\omega^2$ is the element of spherical angle.

We intend to give a different expression of *the same* metric g_{FL} , which is static. Since we know that there is no exact analytical solution in general, we will express it as a Taylor development in the small quantity ϵ . We do not demand that it is global, i.e., that its validity extends to the whole space-time (there would be no solution), but only to a sufficiently extended neighbourhood of the observer today, defined as the event $E_0 = (\text{here, today}) \equiv \{\tau = \tau_o, x = \theta = \phi = 0\}$.

It is convenient to start from the *locally inertial* (non static) form relative to E_0 , defined after defining $\rho := R(\tau_o) x = R_o x$, as

$$g_{\text{FL}} = d\tau^2 - \frac{R^2(\tau)}{R^2(\tau_o)} \left(d\rho^2 + R^2(\tau_o) f_k^2 \left(\frac{\rho}{R(\tau_o)} \right) d\omega^2 \right). \quad (3)$$

The slight change with respect to Eq. (2) emphasizes that, rigorously, an inertial metric must have the Minkowskian form, and that, in Eq. (2), x is an *angular* coordinate.

2.1. The de Sitter case

For pedagogical reasons, and because some exact solutions can be found, we first examine the case of the de Sitter Universe. As usual, we describe de Sitter space-time as the hyperboloid \mathcal{H} isometrically embedded in 5-dimensional Minkowski space-time \mathbb{R}^{1+4} . It is invariant under the de Sitter group $\text{SO}(1, 4)$. Its constant curvature radius is λ^{-1} . Here, $3\lambda^2 = \Lambda$, the true (constant) cosmological constant.

We start from the usual (RW) form of the expanding ($k = -1$) metric:

$$g_{\text{dS}} = d\tau^2 - \frac{\sinh^2 \lambda\tau}{\lambda^2} (d\alpha^2 + \sinh^2 \alpha d\omega^2). \quad (4)$$

The scale factor is $\sinh(\lambda\tau)/\lambda$ and the Hubble parameter $H(\tau) = \lambda \coth(\lambda\tau)$. We define the generalized density

parameter (which, here, includes the only contribution of the cosmological constant) at time τ as $\Omega = \Omega(\tau) = \Omega_\Lambda(\tau) = \lambda^2/H(\tau)^2$ (we recall that λ is a constant). The Einstein equations lead to

$$H^2(\tau) - \frac{\lambda^2}{\sinh^2 \lambda\tau} = \lambda^2 = H^2(\tau) \Omega_\Lambda(\tau). \quad (5)$$

To obtain a static expression, we will make two successive changes of variables. The first one, $(\tau, \alpha) \mapsto (\tau, r)$ with $r \equiv \frac{\sinh \lambda\tau}{\lambda} \sinh \alpha$, gives

$$g_{\text{dS}} = d\tau^2 - \frac{[dr - r H(\tau) d\tau]^2}{1 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}} - r^2 d\omega^2. \quad (6)$$

We now search for a new change of variables $(\tau, r) \mapsto (t, r)$ which suppresses the cross term, i.e., provides a synchronous form of the metric. To do so, we set $\tau = h(t, r)$, which implies $d\tau = \frac{\partial h}{\partial t} dt + h'(t, r) dr$. This requires

$$h'(t, r) \equiv \frac{\partial h}{\partial r} = \frac{-r H}{1 - r^2 H^2 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}}, \quad (7)$$

leading to the exact metric

$$g_{\text{dS}} = \frac{\left(1 - r^2 H^2 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}\right) \left(\frac{\partial h}{\partial t}\right)^2 dt^2}{1 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}} - \frac{1}{1 - r^2 H^2 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}} dr^2 - r^2 d\omega^2. \quad (8)$$

The latter can be simplified thanks to Eq. (5):

$$g_{\text{dS}} = \frac{1 - r^2 \lambda^2}{1 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}} \left(\frac{\partial h}{\partial t}\right)^2 dt^2 - \frac{1}{1 - r^2 \lambda^2} dr^2 - r^2 d\omega^2. \quad (9)$$

2.1.1. The generalized local Birkhoff form

The integration of the differential equation

$$h'(t, r) = -\frac{rH}{1 - r^2 \lambda^2} = \frac{-r \lambda \coth[\lambda h(t, r)]}{1 - r^2 \lambda^2}, \quad (10)$$

with the limiting condition $h(t, 0) = t$, gives

$$h(t, r) = \frac{1}{\lambda} \operatorname{argcosh}[\coth(\lambda t) \sqrt{1 - r^2 \lambda^2}], \text{ and } \left(\frac{\partial h}{\partial t}\right)^2 = 1 + \frac{r^2 \lambda^2}{\sinh^2 \lambda\tau}.$$

Finally, we obtain the desired change of variables

$$(\tau, \alpha) \mapsto (t, r); \quad r = \sinh(\lambda\tau) \sinh \alpha / \lambda, \\ \tanh(\lambda t) = \cosh(\alpha) \tanh(\lambda\tau), \quad (11)$$

and the *local static Birkhoff form* of the (same) metric

$$g_{\text{dS}} = (1 - r^2 \lambda^2) dt^2 - \frac{1}{1 - r^2 \lambda^2} dr^2 - r^2 d\omega^2. \quad (12)$$

This the exact de Sitter metric. Note however that this expression is only valid in a local map, namely inside the causal diamond of the observer (which is sufficient for any local problem). In the next section, we will apply a similar procedure to the general FL model. This will lead to a comparable form, although only exact at order $(H_0 r)^2$.

Geometrically, this may be seen as a projection, in \mathbf{R}^{1+4} , of the hyperboloid. This projection, orthogonal to e_1, e_2 et e_3 , with center 0 in the plane e_0, e_4 , maps \mathcal{H} to the ‘‘hyperbolic nap’’, which is a cylindric hypersurface with hyperbolic section.

We recall that the (local) Birkhoff theorem states that, for any metric with spherical symmetry, in a vacuum universe, there is a local coordinates system (τ, ρ, ω) (corresponding to a peculiar observer) such that the metric coefficients do not depend on time. In a de Sitter universe, the theorem leads to the metric above. For such a form of the metric, calculating the geodesics is particularly simple.

2.2. The general Friedmann-Lemaître model

We now apply the same approach to the general FL model, characterized by a scale factor $R(\tau)$, τ being the cosmic time. We have the Hubble parameter $H(\tau) \equiv \dot{R}(\tau)/R(\tau)$, the deceleration parameter $q(\tau) \equiv -\frac{\ddot{R}}{R^2}$, that we assume negative (corresponding to an accelerating expansion; this is essential for the following calculations), and the Einstein equation

$$H(\tau)^2 + \frac{k}{R(\tau)^2} = \frac{8\pi G\rho(\tau)}{3} \equiv \Omega(\tau) H^2(\tau). \quad (13)$$

We have included in Ω the contributions of matter and the cosmological constant (or exotic energy). *Present* values of these quantities are written with a zero index: $\Omega(\tau_0) = \Omega_0$, etc. Recent observations favour $\Omega_0 \approx 1$, $q_0 \leq -0.55$.

We start from the usual form (3) of the metric, which is expressed in the locally inertial frame for the event $E_0 =$ (today, here). The first change of variables $(\tau, \rho) \mapsto (\tau, r \equiv R(\tau) f_k[\frac{\rho}{R(\tau_0)}])$ leads to the form

$$g_{\text{FL}} = d\tau^2 - \frac{[dr - r H(\tau) d\tau]^2}{1 + [1 - \Omega(\tau)] H^2(\tau) r^2} - r^2 d\omega^2. \quad (14)$$

A comoving galaxy is defined by $d\rho = 0$, which implies $dr = r H(\tau) d\tau$. This formulation of the Hubble law remains exact at any time.

To continue, we define as above a new variable t by $\tau = h(t, r)$ and eliminate the cross term. This requires

$$h'(t, r) \equiv \frac{\partial h}{\partial r} = \frac{-r H[h(t, r)]}{1 - \Omega(\tau) H^2(\tau) r^2}, \quad (15)$$

which allows, using Eq. (13), to write

$$g_{\text{FL}} = \frac{1 - \Omega H^2 r^2}{1 + (1 - \Omega) H^2 r^2} \left(\frac{\partial h}{\partial t} \right)^2 dt^2 - \frac{1}{1 - \Omega H^2 r^2} dr^2 - r^2 d\omega^2, \quad (16)$$

where H and Ω are to be seen as functions $H[h(t, r)]$, $\Omega[h(t, r)]$. We call Eq. (16) the *generalized Birkhoff form* of the model. It would be convenient to define the change of variables $\tau = h(t, r)$, by solving the differential equation

$$h'(t, r) = -\frac{rH[h(t, r)]}{1 - \Omega[h(t, r)] H^2[h(t, r)] r^2}, \quad h(t, 0) = t.$$

This would however require the explicit knowledge of $R(\tau)$, which is not possible analytically, in general. However, we are interested in *local* solutions: we obtain a development as

$$h(t, r) = t - \frac{H(t)}{2H_0^2} (H_0 r)^2 - \frac{H^3(t)}{H_0^4} \frac{[2\Omega(t) + q(t) + 1]}{8} (H_0 r)^4 + O(5), \quad (17)$$

where $O(5)$ means terms of order at least 5 in $(H_0 r)$. Using Eq. (16) above leads to the following local development form of the metric, at the vicinity of the event $E_0 \equiv (t = t_0, r = 0)$:

$$g_{\text{FL}} = \left(1 + q(t) H^2(t) r^2 + [\Omega(t) + q(t)] O(4) \right) dt^2 - \frac{dr^2}{1 - \Omega(t) H^2(t) r^2} - r^2 d\omega^2, \quad (18)$$

where $H(t)$, $\Omega(t)$, etc. are functions of t only.

This form (18) is *non static*, since ΩH^2 depends on time, except for the de Sitter models where $\Omega H^2 = -q H^2 = \lambda^2$. To obtain the static approximation, we have to develop all quantities with respect to $\delta t = t - t_0$, the present time. For instance, $H(t) = H_0 [1 + (q - 1) H_0 \delta t + O(H_0 \delta t)^2]$. However, all time-dependent terms occur in a factor of r^2 (or higher order terms). This allows us, to second order, to replace all time-dependent quantities by their present values, for instance $H(t)$ by H_0 , and similarly for Ω_0 and q_0 . This leads to the approximate form

$$G_{\text{FL}} = [1 + q_0 (H_0 r)^2] dt^2 - [1 - \Omega_0 (H_0 r)^2]^{-1} dr^2 - r^2 d\omega^2. \quad (19)$$

- This form G_{FL} is static and we call it the *local static osculating metric*.
- It implies that the global (cosmological) space-time curvature imprints weak, but non necessarily negligible, effects to the local dynamics, through the three cosmological parameters H_0, q_0, Ω_0 .
- Locally, and for weak fields, the variable r has the meaning of a (spatial) curvature radius, and thus of a distance.
- The expression (19) is at order two. However, it identifies with the exact static form (12) for the de Sitter model. This makes the latter appear as the osculating space-time to the Friedmann-Lemaître model, its second order approximation (the first order one being the tangent Minkowskian space). (We recall that, for an accelerating model of the Universe, q_0 is negative.)
- In the case of a de Sitter model, $\Omega H^2 = \lambda^2 = \text{Cte}$. Then Eq. (19) becomes the exact expression (12).
- Since data indicate $\Omega_0 \approx 1$, $H_0^{-1} \approx 4.3$ Gpc, the metric above gives very good precision $\approx (r/4 \text{ Gpc})^3$.

2.3. Geodesics and redshifts

We derive in Appendix A (29) the geodesic equation, at second order, for a static metric. The explicit form of the metric Eq. (19) gives (at second order)

$$\gamma = H_0^2 r [q_0 + V_i^2 (\Omega_0 - 2q_0)]. \quad (20)$$

This is the radial three-acceleration of a particle in free fall, expressed in the local frame. It is of pure cosmological origin. Note that second order terms are zero.

A ‘‘comoving’’ galaxy is defined by $\rho = \rho_0 = \text{Cte}$. Writing this relation with the new coordinates, we obtain the Hubble law $V = H r$, valid at second order.

3. The local system

3.1. The metric

The previous calculations provided a static approximation of the cosmological metric g_{FL} , exact at second order in $H_0 r$ around a local observer. This was a first step in dealing with the more interesting problem of a massive system in the cosmological context.

The problem is to find the Universe metric G , which is the approximation at first order in ϕ_{Nw} , and at second order in $H_0 r$, of the exact solution of the Einstein equations g . At the lowest (zero) order in $H_0 r$, it describes Newtonian physics. At zero order in ϕ_{Nw} (or, equivalently, when we put $M = 0$), G is equivalent to g_{FL} , up to second order in $H_0 r$.

In the general case, it is not possible to find such a metric analytically. This is possible in the de Sitter case, as it appears below. However, local studies do not require an exact knowledge of this metric: a development at the lowest orders in $H_0 r$ is sufficient. What order is needed, and what is the right development? We answer these two questions below.

We assume a local mass overdensity M with spherical symmetry, isolated in cosmological space, intended to represent the Sun, a galaxy or a cluster, taken as the origin of the coordinates. The gravitational field is assumed weak and all calculations are at first (Newtonian) order only. This excludes the account of any post-Newtonian effect.

We adopt the coordinate system which generalizes that of Eq. (19). The solution of the Einstein equation (with a source) is given, at second order in $H_0 r$, by MAPLE as

$$g = \left(1 - 2 \frac{M}{r} + q_0 (H_0 r)^2 + \mathcal{O}(\epsilon^3)\right) dt^2 - dr^2 \left(1 - 2 \frac{M}{r} - \Omega_0 (H_0 r)^2 + \mathcal{O}(\epsilon^3)\right)^{-1} - r^2 d\omega^2, \quad (21)$$

which reduces to the cosmological form (18) when no mass is present.

The static approximation

$$g \approx G \equiv \left(1 - 2 \frac{M}{r} + q_0 (H_0 r)^2\right) dt^2 - dr^2 \left(1 - 2 \frac{M}{r} - \Omega_0 (H_0 r)^2\right)^{-1} - r^2 d\omega^2 \quad (22)$$

is at order 2. It guarantees that cosmological effects are correctly taken into account at this order above, and that local effects are treated at the Newtonian level. This formula is exact when the cosmological background is de Sitter the well-known Schwarzschild-de Sitter metric; see, e.g., Rindler (2001).

This is to be contrasted with the pure Newtonian approximation:

$$g_{\text{Nw}} = \left(1 - 2 \frac{M}{r}\right) dt^2 - dr^2 \left(1 - 2 \frac{M}{r}\right)^{-1} - r^2 d\omega^2, \quad (23)$$

which is at the same order, but does not take any cosmological effect into account. If we want to compare the two forms, we must be careful not to confuse real effects with coordinate effects. This requires us to consider only covariant quantities, which may however involve the observer.

3.2. The attraction radius

A first application is the definition of the concept of *attraction radius* by a local overdensity in a non-empty environment (Souriau 1975; Mizony 2003): playing the role of a kind of cosmological tidal radius, it is defined as the distance r_M such that the Newtonian contribution of the overdensity becomes equal and opposite to that of cosmological origin: $r_M^3 = \frac{M}{q_0 H_0^2}$. This gives a rough estimate of the ratio of the influences of the two contributions which go like the local / cosmological $\approx (r_M/r)^3$.

The attraction radius due to the gravitational influence of the Sun amounts to ≈ 240 pc. Thus, cosmological effects are expected of the order $(r/240 \text{ pc})^3$, much smaller than the Newtonian ones (see below). For a typical galaxy, $r_M \approx 1$ Mpc. For a cluster like Virgo, $r_M \approx 20$ Mpc indicates that cosmological effects may become not completely negligible, and a fortiori for superclusters.

3.3. Dynamics in the local system

3.3.1. Radial or tangential free fall

The geodesic equations in a static metric are derived in Appendix A. Replacing the metric coefficients by their values (22), we obtain the radial geodesic equation

$$r(t) = r_o + v_o(t - t_o) + \frac{\gamma(r_o)}{2}(t - t_o)^2 + \dots,$$

with

$$\gamma(r) = -\frac{M}{r^2} - q_0 H_0^2 r + (2q_0 + \Omega_0) H_0^2 M + \left(\frac{3M}{r^2} + (2q_0 - \Omega_0) H_0^2 (r + M)\right) V^2(r), \quad (24)$$

to be compared to

$$\gamma_{\text{Nw}} = -\frac{M}{r^2} + \left(\frac{3M}{r^2}\right) V^2(r).$$

For tangential free fall, the application of the formulae above to Eq. (31) gives

$$r V_\theta^2 = \gamma_\theta = \frac{M}{r^2} + q_0 H_0^2 r, \quad (25)$$

i.e., a contribution similar to the radial case.

3.3.2. Observations

The motions of a probe in the Solar system, a star in a galaxy, a galaxy in a cluster, etc. are not directly observable. The only information available to the terrestrial observer is the redshift, that can be monitored as a function of his proper time. Such dynamical analyses are at the basis of the Pioneer effect, and of the estimations of dark matter in the astronomical structures. An exact estimation of the cosmological effects therefore appears very important.

The most direct effect is the addition of a cosmological component to the Newtonian redshift (i.e., velocity, through the Doppler effect). This may affect, for instance, the dark matter estimations from rotation or dispersion curves. Also, an additional term of cosmological origin is added in the derivative of z (w.r.t. the observer proper time). Precisely, the Pioneer effect appears in the form of such an additional term, with the dimension of an acceleration. However, the calculation below clearly shows that, contrary to recent claims, it is definitely not of cosmological origin.

The Pioneer effect is not cosmological

The inertial probe emits a light-ray at (t_s, r_s) . It is received by the observer at $(t = T, r = 0)$. Relativistic optics (Eq. (32)) relates T (the time when the observer receives the light) to t_s . The observer monitors the redshift as a function $z(T)$ of the observer's proper time, calculated in Appendix B. The observer may calculate the derivative $a \equiv \frac{dz}{dT}$.

We recall the Pioneer effect (Anderson et al. 1998; Turyshev et al. 2004) as a time variation of the observed redshift of the Pioneer probe, reported as an anomalous acceleration

$$a_{\text{Pioneer, observed}} = (8.74 \pm 1.33) \times 10^{-10} \text{ m s}^{-2}$$

towards the sun, at a distance $20 \text{ a.u.} \approx 10^{-5} \text{ pc} \approx 4.5 \times 10^{16} \text{ m}$ from it.

We applied the calculations of Appendix B to the Pioneer probe, a source in radial free fall in the Solar System. This gives a cosmological contribution $a_{\text{Pioneer, cosmological}} = q_0 H_0^2 r$.

Numerical estimations (with $h_0 = .7$) give $5q_0 10^{-20} \text{ m s}^{-2}$, about 10^{10} orders of magnitude smaller than the observed effect.

3.3.3. Cosmological effects in extragalactic astronomy

Evaluations of dark matter in galaxies or clusters result from redshift measurements of particles (stars, gas molecules, galaxies) in free fall in the gravitational potential. The formula (25) is directly applicable to evaluate the contribution of cosmology.

At the periphery of a galaxy (30 kpc), the cosmological acceleration reaches $a \approx 4 \times 10^{-15} \text{ m s}^{-2}$. This corresponds to a velocity component of about 2 km s^{-1} , to be compared to a typical Newtonian rotation velocity of several 100 km s^{-1} .

At the periphery of a cluster of galaxies (5 Mpc), the cosmological acceleration reaches $a \approx 6 \times 10^{-13} \text{ m s}^{-2}$. The additional velocity, about 300 km s^{-1} , may be non-negligible in precise estimations of dark matter. Its contribution increases with

the size of the system, and thus becomes important for estimations of deceleration or acceleration beyond the cluster size. In particular, the metric G is perfectly adapted to explore the large-scale velocity fields (like the Virgocentric flow), where cosmological and local effects are of the same orders of magnitude: it appears much more convenient than the Tolman-Bondi expression. Such analyses are in progress.

In these situations, cosmology affects not only the dynamics, but also the gravitational optics (lensing of light rays). The importance of the latter, as astrophysical and cosmological tools, justifies the precise estimation of these effects thanks to the metric G . Those will be estimated in a forthcoming paper.

4. Conclusions

The metric of a given space-time can be written in multiple forms. Each one may offer a peculiar geometrical and/or physical interest. Each one has its own validity range, in general lower than the whole space-time manifold, as is well shown by the isometric embedding in \mathbf{R}^5 (Lachièze-Rey 2001; Mizony 2003).

Here we have provided a local static form (19) for the pure cosmological metric. Although its use may be highly convenient for a local cosmological study, its interest remains rather academic, since an exact (although non-static) form is available. However, this emphasizes the fact that the local cosmology (in a wide sense) is that of the de Sitter space-time, which appears as the osculating space-time to any FL cosmology.

Addressing the question of a local overdensity in the cosmological context, we have found a static solution (22), which is exact at order $(H_0 r)^2$, largely sufficient for any study in the vicinity of a local overdensity in the Universe. This clearly expresses the influence of cosmology in such situations, for gravitational dynamics and optics.

We have evaluated precisely the cosmological effects in the solar system. They appear negligible, which implies unambiguously that the unexplained Pioneer acceleration is not of cosmological origin. We have estimated such effects for dynamical studies of galaxies, clusters, superclusters etc., and in particular how they may affect dark matter estimations.

Our proposed metric form G offers a perfect framework to study any situation where cosmological effects are not negligible compared to the local ones. This is the case for large-scale velocity fields like the local Virgocentric flow. This is also the case for gravitational optics at the periphery of large scale mass condensations. This powerful tool for estimations of (dark) massive matter cannot be used without correct estimations of the cosmological effects, which may become non-negligible (work in progress).

The metric G offers an unique framework for the exploration of the extragalactic Universe, where cosmological effects superpose on the local ones.

Appendix A: Geodesic equations

Inertial motion is given by the geodesic equations. We give a complete derivation, which allows an adaptation to more

complicated situations. We derive the geodesic equations as the Euler-Lagrange equations for the Lagrangian $\mathcal{L} = 1/2 g_{\mu\nu} v^\mu v^\nu$, with $v^\nu \equiv \frac{dx^\nu}{d\lambda}$, and λ is a time-parameter:

$$\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\rho} v^\mu v^\nu = \frac{d(g_{\mu\rho} v^\mu)}{d\lambda}. \quad (26)$$

Here, the coefficients of the metric reduce to the diagonal ones, and depend on the coordinate r only (static form). The first equation ($\rho = t$) implies that $v_t \equiv v^t g_{tt}$ is a constant of motion. This is valid with any choice of the parameter λ . In particular, for $\lambda = \tau$ (the proper time), this equation gives $u_\tau = E = \text{Cte}$, which expresses energy conservation by the motion.

Radial motion

For radial time-like geodesics, the second one ($\rho = r$) is equivalent to the unitarity condition for the four velocity u . In our case, it is more convenient to work with the latter, namely

$$g_{tt} (u^t)^2 + g_{rr} (u^r)^2 = (u_t)^2/g_{tt} + g_{rr} (u^r)^2 = 1. \quad (27)$$

Defining a 3-velocity $V \equiv \frac{dr}{dt} = \frac{u^r}{u^t}$, this can be rewritten $g_{tt} + g_{rr} V^2 = (\frac{g_{tt}}{u_t})^2 = (\frac{g_{tt}}{E})^2$, that we simply derive w.r.t. t . This defines a 3-acceleration $\gamma \equiv \frac{d^2r}{dt^2}$, and we obtain

$$\gamma = \left(\frac{2g_{tt} g'_{tt}}{u_t^2} - g'_{rr} V^2 - g'_{tt} \right) / (2g_{rr}),$$

where the prime means the derivative w.r.t. r . Taking into account initial conditions,

$$g_{rr}(r_i) V_i^2 = g_{tt}^2(r_i)/(u_t)^2 - g_{tt}(r_i), \quad (28)$$

and developing, we obtain the radial geodesic equation at second order,

$$r(t) = r_i + V_i(t - t_i) + \frac{\gamma(r_i)}{2}(t - t_i)^2 + \dots, \quad (29)$$

with

$$\gamma(r_i) = \left(\frac{1}{2g_{rr}} \frac{dg_{tt}}{dr} + \frac{V_i^2}{g_{tt}} \frac{dg_{tt}}{dr} - \frac{V_i^2}{2g_{rr}} \frac{dg_{rr}}{dr} \right) (r_i) \quad (30)$$

appearing as the radial three-acceleration.

Tangential motion

Tangential motion is defined by $u^r = 0$. The geodesic equations imply that all components of the velocity remain constant. Moreover, the second equation reads

$$g'_{tt} u^t u^t + g'_{\theta\theta} u^\theta u^\theta.$$

Joined to the unitarity of the velocity, and defining

$$V_\theta \equiv \frac{d\theta}{dt} = \frac{u^\theta}{u^t},$$

this gives

$$r V_\theta^2 = \frac{g'_{tt}}{2}, \quad (31)$$

where the RHS appears as the centripetal acceleration.

Calculating the redshift as a Doppler one, we have $z \approx V \approx \frac{M}{r} + q H^2 r^2$. This means that we have an additional term of cosmological origin when we measure a rotation curve in a galaxy, for instance. A numerical estimation gives the additional cosmic contribution as $\Delta V^2 = q H^2 r^2 \approx 4q_0(\text{km s}^{-1})^2$, for $r = 30 \text{ kpc}$.

Appendix B: Redshift in a static metric

We assume a static metric $ds^2 = A^2(r) dt^2 - B^2(r) dr^2$. We calculate the redshift (seen by the observer at the origin of coordinates, $r = 0$) emitted by a (moving) source in radial motion, $r = f(t)$, thus with a radial 3-velocity $V_s \equiv df/dt$. We assume the light ray is also radial. Calculations are at second order in $H_0 r$.

The light ray, emitted by the source at (t_s, r_s) , reaches the observer at $(T, r = 0)$:

$$T - t_s = \int_0^{r_s} \frac{B(r) dr}{A(r)}.$$

A short delay after, the source has moved to $(t_s + \delta t_s, r_s + \delta r_s)$, and emits a second light-ray, which reaches the observer at $(T + \delta T, r = 0)$:

$$T - t_s + \delta T - \delta t_s = \int_0^{r_s + \delta r_s} B(r)/A(r) dr. \quad (32)$$

Subtraction gives

$$\delta T - \delta t_s = \int_{r_s}^{r_s + \delta r_s} \frac{B(r)}{A(r)} dr \approx \frac{B(r_s)}{A(r_s)} \delta r_s.$$

Considering the two rays as two light fronts, as usual, we obtain the two periods of emission and reception:

$T_{\text{emission}} = [A(r_s)^2 \delta t_s^2 - B(r_s)^2 \delta r_s^2]^{1/2}$ (in proper time of the source),

$T_{\text{reception}} = \delta T$, since T measures the proper time of the observer. Thus (definition), the redshift: $1 + z \equiv \frac{T_{\text{reception}}}{T_{\text{emission}}} = \frac{\delta T}{[A(r_s)^2 \delta t_s^2 - B(r_s)^2 \delta r_s^2]^{1/2}}$. Combining, with $\delta r_s = V_s \delta t_s$, some algebra leads to

$$1 + z = \frac{B_s V_s + A_s}{A_s [A_s^2 - B_s^2 V_s^2]^{1/2}} = \frac{1}{A_s} \sqrt{\frac{[B_s V_s + A_s]}{[A_s - B_s V_s]}}. \quad (33)$$

Writing for simplification, $A_s = 1 + a$ and $B_s = 1 + b$, with a and b second order quantities, we obtain the development

$$1 + z \approx \left[1 - a + V_s \frac{b - a}{1 - V_s^2} \right] \sqrt{\frac{1 + V_s}{1 - V_s}}. \quad (34)$$

When the source is non relativistic ($V_s \ll 1$), this reduces to

$$z \approx V_s (1 + b - a) - a.$$

All subsequent calculations will be at first order in V_s .

Specifying the metric as in the text, this becomes

$$z \approx V_s \left[1 + \frac{2M}{r} + (\Omega - q) \frac{H^2 r^2}{2} \right] + \frac{M}{r} - q \frac{H^2 r^2}{2}. \quad (35)$$

This is the redshift measured by the observer, that can be recorded as a function of his proper time T_1 . Calculating the derivative, the observer will obtain a three-acceleration $a \equiv \frac{dz}{dT_1} = \frac{dz}{dt_s} / \frac{dT_1}{dt_s}$, where the last derivative is calculated from Eq. (32) as $1 + \frac{V_s B(r_s)}{A(r_s)} \approx 1 + V_s(1 - a + b)$.

Radial inertial source

To calculate $\frac{dz}{dt_s}$, we assume the source in radial inertial motion (29). From Eq. (35),

$$\frac{dz}{dt_s} = \gamma(r_s) \left[1 + 2\frac{M}{r} + (\Omega - q) H^2 \frac{r^2}{2} \right] - \frac{V_s}{r} \left(\frac{M}{r} + q H^2 r^2 \right),$$

where we have neglected quadratic terms in the velocity. Inserting $\gamma(r_s) \approx -a' (1 - 2b)$, and $a' = \frac{M}{r^2} + q H^2 r$, we obtain (at second order)

$$\frac{dz}{dt_s} \approx -\left(\frac{M}{r^2} + q H^2 r \right):$$

we observe a cosmic three-acceleration $-q H^2 r$ in addition to the Newtonian one.

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