A&A 431, 415–421 (2005)
DOI: 10.1051/0004-6361:20041472

Astronomy & Astrophysics

C^∞ perturbations of FRW models with a cosmological constant

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Received 15 June 2004 / Accepted 8 October 2004

Abstract. Spatially homogeneous and isotropic cosmological models, with a perfect fluid matter source and non-vanishing cosmological constant, are studied. The equations governing linear perturbations of the space-time and the variation of energy density are given. The complete solution of the problem is obtained for C^∞ perturbations, using a comoving time. The Sachs-Wolfe fluctuations of the temperature of the cosmic background radiation are obtained for the relatively growing density perturbations. It is found that the observable celestial microwave fluctuation pattern underwent a reversal approximately two billion years ago. What is observed today is a negative image of the last scattering surface with an attenuation of the fluctuations, due to the presence of the cosmological constant.

Key words. cosmology: cosmic microwave background – cosmology: large scale structure of Universe

1. Introduction

Recent observations by the *High-z Supernova Search Team* (Tonry et al. 2003) corroborate the data on an accelerating expansion of the Universe. Three independent lines of evidence [those from the *Wilkinson Microwave Anisotropy Probe* (*WMAP*) measurements (Spergel et al. 2003) of the cosmic microwave background radiation *(CMBR)*, the *Sloan Digital Sky Survey* (*SDSS*) (Tegmark et al. 2004a) and the observations of type-Ia supernova spectra] converge on the value $\Omega_\Lambda = 0.70 \pm 0.04$ of dark energy. All data are consistent, within a 20% error bar, with a time-independent dark energy distribution, as is the case with the cosmological constant Λ and flat space ($k = 0$). These developments attracted attention to cosmological models in the presence of a Λ term in Einstein's gravitational equations. Many excellent review papers are available (Carroll 2001; Peebles & Ratra 2002; Carroll et al. 1992; White et al. 1994) on the implications of a cosmological constant in models of the Universe. While the cosmological constant has little effect on the large-scale structure and dynamics of the Universe, the microwave fluctuations will be significantly different in $\Lambda \neq 0$ models (White et al. 1994). The scalar perturbations of $\Lambda \neq 0$ models with multicomponent fluids (i.e., cold or hot dark matter, photons and massless neutrino) have been computed by Bond & Efstathiou (1987), Fukugita et al. (1990), Holtzman (1989), Vittorio et al. (1991), Hu & Sugiyama (1995) and Stompor (1994) in the linear approximation. These numerical results have been compared by Stompor (1994) and found to be in reasonable agreement. Hu & Sugiyama (1995) claim that for $\Lambda \neq 0$, one must use a numerical approach. In our work, however, we present an analytic treatment of the collisionless dust which represents the late-time evolution.

A complete solution describing the effect of the cosmological constant on perturbation dynamics is lacking in the literature, this including the rotational and gravitational wave perturbations. The temperature fluctuations of the CMBR have been computed in numerical schemes (Bond & Efstathiou 1987; Fukugita et al. 1990; Holtzman 1989; Vittorio et al. 1991; Stompor 1994) and Multamaki & Elgaroy (2003) investigate the integrated contributions. Known as the Sachs-Wolfe effect, this is the variation in the redshift of a photon travelling freely in the universe rippled with perturbations. Initial work, both on the zero-pressure model of the current state of the Universe, and on the radiation-dominated era, was done by Sachs & Wolfe (1967). The pressure-free universe in the presence of a cosmological constant has been considered by Heath (1977) and Lahav et al. (1991). They used the redshift parameter to investigate the behaviour of the density contrast. A more recent work by Vale & Lemos (2001) uses the formalism of Padmanabhan (1993) to compute linear perturbations of a dust-filled Friedmann-Robertson-Walker (FRW) universe in the presence of a cosmological constant. They find that the presence of Λ inhibits the growth of the fluctuations in time. (A different approach to the perturbation problem of a different (de Sitter) model, using the gauge-independent formulation, has been pursued by Barrow (2003), for the investigation of stability under perturbations.)

Neither of these earlier works reaches a complete perturbative picture. Although the Sachs-Wolfe effect is computed by numerical methods, there is no analytic treatment so far available for $\Lambda \neq 0$. Our aim here, therefore, is to consider, in an analytic framework, the Sachs-Wolfe effect and its contributions to the fluctuations of the CMBR in the presence of a cosmological constant. We follow the treatment of the perturbations due to White (1973), thereby relaxing the momentum conditions of the original Sachs-Wolfe work. Both works yield all C^∞ perturbations although this was only shown by White (1973). Correspondingly, in the present paper we also obtain all C^∞ perturbations. In Sect. 2, we enlist the perturbed field equations using the conformal time η. While retaining the form of the metric functions, we change to the comoving time t in the field equations so obtained. The reason for this is that the unperturbed radius a has a simple analytic dependence on t.

The general solution of the linear field equations in a dust-filled universe is obtained in Sect. 3. We find that all first-order fields, except the wave solutions, can be expressed in terms of only two complex incomplete elliptic integrals E and F. This is achieved by using the mirror symmetries of the elliptic integrals.

In Sect. 4, we obtain the Sachs-Wolfe effect and its integral contribution which vanishes in the $\Lambda \to 0$ limit. An unexpected feature of the transfer function is that it changes sign and reverses the pattern of the temperature fluctuation on the celestial sphere at late times. In Sect. 5, we investigate the physical interpretation of our solution. Taking the current experimental values of the cosmological parameters, we obtain the comoving time elapsed between the emission at the surface of last scattering and reception of the photon. We find that the Sachs-Wolfe contribution to the CMB fluctuations in the presence of the cosmological constant is damped by a factor of 3 today, because of the proximity, on the cosmological time scale, of the moment of reversal of the microwave temperature fluctuations and the reception. This feature of the SW effect has remained concealed in earlier works, because of the numerical methods involved.

2. Linear perturbations

In this section we present the equations of a perturbed spatially flat ($k = 0$) FRW cosmology in the linear approximation. The metric is that of a perturbed FRW model in the conformal form

$$g_{ab} = a^2(\eta)(\eta_{ab} + h_{ab}), \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$, h_{ab} is the metric perturbation and the scale function $a = a(\eta)$ is determined by the field equations. Roman indices run from 0 to 3 and Greek indices from 1 to 3. Except when we compute numerical values, we employ units such that the speed of light is $c = 1$ and the gravitational constant $G = 1/8\pi$. We use the Minkowski metric η_{ab} and its inverse to raise and lower indices of h_{ab} and of other small quantities.

The matter source is assumed to be a perfect fluid,

$$T^{ab} = (\tilde{\rho} + \tilde{p})u^a u^b - \tilde{p} g^{ab} \tag{2}$$

with the four velocity u^a of the fluid normalized by $u^i u_i = 1$. The cosmological constant Λ of the Einstein equations is introduced in the energy-momentum tensor. The effective energy density $\tilde{\rho}$ and pressure $\tilde{p}$ are related to the corresponding fluid quantities as

$$\tilde{\rho} = \rho + \Lambda\,, \quad \tilde{p} = p - \Lambda. \tag{3}$$

In the perturbed space-time we choose comoving coordinates (Sachs & Wolfe 1967). Using this gauge, the four velocity of the fluid and the metric perturbation h_{00} are

$$u^a = \frac{\delta^a_{\ 0}}{a}\,, \quad h_{00} = 0. \tag{4}$$

The coordinate transformations $x^a \to x^a - \xi^a$, which preserve the coordinate condition, have the following properties

$$\xi^r u^a_{\ ,r} - u^r \xi^a_{\ ,r} = 0 \ \Rightarrow\ \xi^0 = \frac{b(x^\alpha)}{a}\,, \quad \xi^\alpha = c^\alpha(x^\beta), \tag{5}$$

where the arbitrary functions b and c^α are independent of the conformal time η.

The conservation of energy-momentum

$$T^{ab} = (\rho + \delta\rho + p + \delta p)u^a u^b - (p + \delta p - \Lambda)g^{ab} \tag{6}$$

implies the unperturbed relations

$$3(\rho + p)a' + a\rho' = 0\,, \quad p_{,\alpha} = 0 \tag{7}$$

and the perturbative equations

$$\delta\rho' + 3\frac{a'}{a}(\delta\rho + \delta p) + (\rho + p)\frac{h'}{2} = 0, \tag{8}$$

$$\left(h^{\alpha 0\,\prime} + \frac{a'}{a}h^{\alpha 0}\right)(\rho + p) + h^{\alpha 0}p' - \delta p_{,}^{\ \alpha} = 0 \tag{9}$$

where $h = \eta^{rs}h_{rs}$. The prime denotes derivative with respect to the conformal time η.

Einstein equations $G_{ab} = T_{ab}$ to leading order give

$$3\frac{a'^2}{a^4} = \rho + \Lambda\,, \quad 2\frac{a''}{a^3} - \frac{a'^2}{a^4} = -p + \Lambda. \tag{10}$$

The perturbed field equations for a perfect fluid have been obtained by White (1973) . They remain valid in the presence of the cosmological constant provided the pressure and density are replaced by their tilded values according to Eq. (3). The scale factor a is now a solution of Eqs. (10). Combining the 00 component and the trace of the spatial part of the Einstein equations we have

$$\begin{aligned} h'' \ &= \ -2\frac{a'}{a}h' + 2h^{0\mu}_{\ \ ,\mu}{}' + 4\frac{a'}{a}h^{0\mu}_{\ \ ,\mu} \\ &\quad +\frac{1}{2}S^{\mu\nu}_{\ \ ,\mu\nu} + \frac{1}{3}\nabla^2 h - 3a^2\delta p, \end{aligned} \tag{11}$$

$$S^{\mu\nu}_{\ \ ,\mu\nu} + \frac{2}{3}\nabla^2 h + 2\frac{a'}{a}\left(2h^{0\mu}_{\ \ ,\mu} - h'\right) + 2a^2\delta\rho = 0. \tag{12}$$

In these equations $S_{\alpha\beta} = h_{\alpha\beta} - \eta_{\alpha\beta}h/3$ denote the trace-free part of the perturbations and $\nabla^2 f = -\eta^{\mu\nu}f_{,\mu\nu}$, where $\nabla^2 \equiv \Delta$ is the standard Laplacian, for an arbitrary smooth function f. The remaining components of the linearized Einstein equations are

$$\nabla^2 h^{0\alpha} - \frac{2}{3}h_{,}^{\ \alpha\prime} + S^{\alpha\mu}_{\ \ ,\mu}{}' + h^{0\mu}_{\ \ ,\mu}{}^{\alpha} + 4\left(\frac{a''}{a} - 2\frac{a'^2}{a^2}\right)h^{0\alpha} = 0 \tag{13}$$

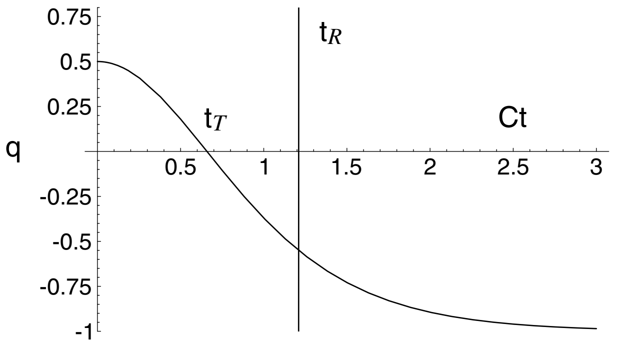

Fig. 1. The deceleration parameter q as a function of the dimensionless comoving time Ct. The present age of the universe is $Ct_R = 1.21$ and the acceleration commences at comoving time $Ct_T = 0.66$ (cf. Sect. 5).

$$\begin{aligned} S^{\alpha}_{\ \beta}{}'' + 2\frac{a'}{a}S^{\alpha}_{\ \beta}{}' - \nabla^2 S^{\alpha}_{\ \beta} &= S^{\alpha\mu}_{\ \ ,\beta\mu} + S_{\beta\mu,}^{\ \ \ \alpha\mu} - \frac{2}{3}\delta^{\alpha}_{\beta}S^{\mu\nu}_{\ \ ,\mu\nu} \\ &+ h^{\alpha0}_{\ \ ,\beta}{}' + h_{\beta0,}^{\ \ \ \alpha}{}' - \frac{1}{3}h_{,\beta}^{\ \ \alpha} + 2\frac{a'}{a}(h^{\alpha0}_{\ \ ,\beta} + h_{\beta0,}^{\ \ \ \alpha}) \\ &- \frac{1}{3}\delta^{\alpha}_{\beta}\left[\frac{1}{3}\nabla^2 h + 2h^{0\mu}_{\ \ ,\mu}{}' + 4\frac{a'}{a}h^{0\mu}_{\ \ ,\mu}\right]. \end{aligned} \tag{14}$$

3. Integration of the field equations

In this section we solve the field equations in the presence of pressureless matter, $p = \delta p = 0$ for perturbations in the class of C^∞ functions. We then repeatedly use the following
Lemma (*Brelot*)
If g is any C^∞ function on E^3, then there exists a C^∞ function f on E^3 such that $\nabla^2 f = g$.
(A modern proof was provided by Friedman 1963). We introduce the comoving time coordinate t by the relation $ad\eta = \mathrm{d}t$. The solution of Eqs. (10) for matter (Stephani et al. 2003) is

$$p = 0, \quad \rho a^3 = C_M, \quad a = a_0 \sinh^{2/3}(Ct + C_0), \tag{15}$$
$$a_0 = \left(\frac{C_M}{\Lambda}\right)^{1/3}, C = \frac{\sqrt{3\Lambda}}{2}, H_0 = \left(\frac{\Lambda}{3}\right)^{1/2}\coth(\sqrt{3\Lambda}t_0),$$

where $C_0 = Ct_0$ and H_0 is the Hubble constant. With a new definition of t we set $C_0 = 0$, and the big bang occurs at $t = 0$.

The value of the deceleration parameter (Misner 1973)

$$q = -\frac{\ddot{a}}{a}\left(\frac{\dot{a}}{a}\right)^{-2} \tag{16}$$

decreases after the big bang, and changes sign (acceleration) at the moment of turnover $t = t_T$. A dot denotes the derivative with respect to the comoving time coordinate t. The deceleration curve is displayed in Fig. 1. In the limit of a vanishing cosmological constant, the turnover time t_T moves to infinity, and the radius has the time dependence $a = a_0 t^{2/3}$.

The functional change of h_{ab} induced by the transformation (5) is expressed by the Lie derivative of the metric g_{ab} with respect to ξ^a,

$$h_{00} \to h_{00}\, , h_{0\alpha} \to h_{0\alpha} + \frac{b_{,\alpha}}{a}\, , h \to h + 2c^{\mu}_{\ ,\mu} + 6b\frac{\dot{a}}{a} \tag{17}$$
$$h_{\alpha\beta} \to h_{\alpha\beta} + c_{\alpha,\beta} + c_{\beta,\alpha} + 2b\frac{\dot{a}}{a}\eta_{\alpha\beta}.$$

Equation (9) is integrated for $h^{0\alpha}$. With a coordinate transformation and using Brelot's Lemma, one can set (White 1973)

$$h^{0\alpha} = \frac{1}{a}\nabla^2 C^\alpha, \tag{18}$$

where $C^\alpha = C^\alpha(x^\beta)$ is a spatial function with

$$C^{\mu}_{\ ,\mu} = 0. \tag{19}$$

As a consequence $h^{0\mu}_{\ \ ,\mu} = 0$ holds. Gauge transformations which preserve the above form of $h^{0\alpha}$ satisfy $\nabla^2 b = 0$.

Integrating Eq. (8) with respect to the conformal time we get

$$h = -2\frac{\delta\rho}{\rho} + H, \tag{20}$$

where $H = H(x^\alpha)$ is an integration function.

From the perturbed field Eqs. (11) and (12) we have

$$h'' = -2\frac{a'}{a}h' + \frac{1}{2}S^{\mu\nu}_{\ \ ,\mu\nu} + \frac{1}{3}\nabla^2 h, \tag{21}$$
$$\delta\rho = \frac{a'}{a^3}h' - \frac{1}{a^2}\left(\frac{1}{2}S^{\mu\nu}_{\ \ ,\mu\nu} + \frac{1}{3}\nabla^2 h\right). \tag{22}$$

From the above expressions

$$h'' + \frac{a'}{a}h' + a^2\delta\rho = 0. \tag{23}$$

Inserting here h from Eq. (20), changing to the comoving time t and using the unperturbed solution (15), the equation for the density perturbation is

$$\delta\ddot{\rho} + 8\frac{\dot{a}}{a}\delta\dot{\rho} + 3\left(\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} - \frac{C_M}{6a^3}\right)\delta\rho = 0. \tag{24}$$

Substituting the field Eqs. (10) we have

$$\delta\ddot{\rho} + 8\sqrt{\frac{\Lambda}{3}}\coth(Ct)\delta\dot{\rho} + \Lambda\left(\frac{3}{\sinh^2(Ct)} + 5\right)\delta\rho = 0. \tag{25}$$

Changing to the variable

$$z = \cosh(2Ct) \tag{26}$$

results in the homogeneous equation

$$3\left(z^2 - 1\right)(z - 1)\frac{\mathrm{d}^2\delta\rho}{\mathrm{d}z^2} + \left(11z^2 - 3z - 8\right)\frac{\mathrm{d}\delta\rho}{\mathrm{d}z} + (5z + 1)\delta\rho = 0. \tag{27}$$

This is a special case of the Heun equation (Kamke 1959). Application of the Kovacic algorithm (Kovacic 1986) yields the solution

$$\delta\rho = \frac{\sqrt{z+1}}{(z-1)^{3/2}}\left[K_1 - K_2\int_1^z \frac{\mathrm{d}z'}{(z'-1)^{1/6}(z'+1)^{3/2}}\right], \tag{28}$$

where $K_1 = K_1(x^\alpha)$ and $K_2 = K_2(x^\alpha)$ are space functions. This contains the elliptic integral

$$I = \int_1^z \frac{\mathrm{d}z'}{(z'-1)^{1/6}(z'+1)^{3/2}} = \frac{3}{2^{2/3}}\int_0^x \left(\frac{x'}{x'^3+1}\right)^{3/2}\mathrm{d}x', \tag{29}$$

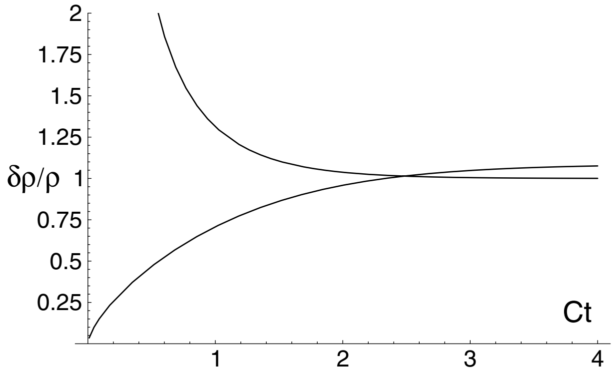

Fig. 2. The relatively decreasing and growing contributions to the density contrast.

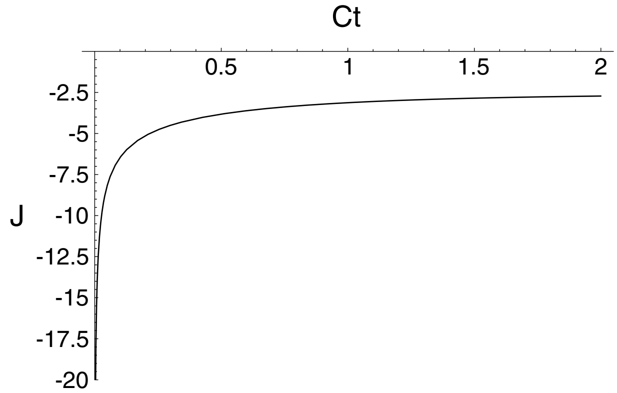

Fig. 3. Time dependence of J.

where we have introduced the new variable x by

$$z = 2x^3 + 1. \tag{30}$$

Hence, using Eq. (20)

$$h = H - \coth(Ct)\,(K_1 - K_2 I)\,. \tag{31}$$

Taking the divergence of Eq. (13), we obtain a total time derivative. By Brelot's Lemma, the integral of this equation defines $\nabla^2 B$ such that

$$\nabla^2 B = \frac{1}{2} S^{\mu\nu}{}_{,\mu\nu} + \frac{1}{3}\nabla^2 h, \tag{32}$$

where $B = B(x^\alpha)$ is a space function. From (11) we have

$$h'' = -2\frac{a'}{a}h' + \nabla^2 B. \tag{33}$$

Substituting here the form (31) of h, the terms proportional to K_1 cancel and we get

$$K_2 = \frac{3\nabla^2 B}{2^{4/3} a_0^2 C^2}\cdot \tag{34}$$

Since K_1 is a C^∞ function, it may be represented (Friedman 1963) by a new spatial function $A(x^\alpha)$ as follows,

$$K_1 = \nabla^2 A. \tag{35}$$

The time dependence of the density contrast $\delta\rho/\rho$ is displayed in Fig. 2. The contributions of the terms

$$\delta_A = \coth(Ct), \qquad \delta_B = \coth(Ct) I$$

proportional to the spatial functions A and B, respectively, are relatively decreasing and growing. They tend to constant values in the asymptotic future,

$$\lim_{t\to\infty} \delta_A = 1, \quad \lim_{t\to\infty} \delta_B = 2^{-2/3} Beta(\tfrac{2}{3}, \tfrac{5}{6}) = 1.086517930,$$

where *Beta* is Euler's integral of the first kind. These terms have the series expansion in powers of $\tau = Ct$

$$\delta_A = \frac{1}{\tau} + \frac{1}{3}\tau - \frac{1}{45}\tau^3 + O(\tau^4), \tag{36}$$
$$\delta_B = 2^{1/3}\frac{3}{5}\tau^{2/3} + O(\tau^{8/3}).$$

In the limit of a vanishing cosmological constant, these are the density perturbations of Sachs & Wolfe (1967).

If we rewrite Eq. (13) by introducing the new variable

$$w = \sinh(Ct)$$

we have

$$\frac{\partial}{\partial w}\left(S^{\alpha\mu}{}_{,\mu} - \frac{2}{3}h_{,}{}^{\alpha}\right) + \frac{\nabla^2\nabla^2 C^\alpha}{a_0^2 C w^{4/3}\sqrt{1+w^2}} - \frac{8C}{3}\frac{\nabla^2 C^\alpha}{w^2\sqrt{1+w^2}} = 0. \tag{37}$$

The integration of Eq. (37) and use of Eq. (31) yields

$$S^{\alpha\mu}{}_{,\mu} + \frac{2}{3}\coth(Ct)\,(K_1 - K_2 I)_{,}{}^{\alpha} + \frac{8C\nabla^2 C^\alpha}{3}\frac{\sqrt{1+w^2}}{w} + \frac{\nabla^2\nabla^2 C^\alpha}{a_0^2 C}\int_0^w \frac{dw'}{w'^{4/3}\left(1+w'^2\right)^{1/2}} + J^\alpha = 0. \tag{38}$$

Here $J^\alpha = J^\alpha(x^\beta)$ are integration functions. From Eq. (17), the trace-free part $S_{\alpha\beta}$ has the transformation properties

$$S_{\alpha\beta} \to S_{\alpha\beta} \;+\; 2c_{(\alpha,\beta)} - \frac{2}{3}\eta_{\alpha\beta}c^\mu{}_{,\mu}. \tag{39}$$

The generator c^α of the gauge transformations can be chosen (White 1973) such that the integration function in Eq. (31) is $H = 3B$ and in Eq. (38) we set $J^\alpha = 0$. The normal form of the elliptic integral

$$J = \int_0^w \frac{\mathrm{d}w'}{w'^{4/3}\left(1+w'^2\right)^{1/2}} \tag{40}$$

will be given in the Appendix. The graph of J is displayed in Fig. 3. The following relation holds between the time variables:

$$x = \frac{a}{a_0} = \sinh^{2/3}(Ct) = \left(\frac{z-1}{2}\right)^{1/3} = w^{2/3}, \tag{41}$$

$$x^3 = \sinh^2(Ct) = \frac{1}{2}(\cosh(2Ct) - 1) = \frac{z-1}{2}, \tag{42}$$

where the ranges are $0 < t < \infty$, $0 < x < \infty$, $0 < w < \infty$, $1 < z < \infty$.

Equation (14) for the trace-free part of the perturbations is

$$a^2\ddot{S}^{\alpha}_{\ \beta} + 3a\dot{a}\dot{S}^{\alpha}_{\ \beta} - \nabla^2 S^{\alpha}_{\ \beta} = h_{,\beta}^{\ \alpha} - 4B_{,\beta}^{\ \alpha} \\ -2C\nabla^2(C^{\alpha}_{\ ,\beta} + C_{\beta,}^{\ \ \alpha})\coth(Ct) + \frac{\delta^{\alpha}_{\beta}}{3}(\nabla^2 h - 4\nabla^2 B) \\ -\frac{\nabla^2\nabla^2(C^{\alpha}_{\ ,\beta} + C_{\beta,}^{\ \ \alpha})}{a_0^2 C}J \tag{43}$$

with h as given in Eq. (31).

A particular solution for $S^{\alpha}_{\ \beta}$ can be constructed by considering the terms with a decreasing number of Laplacians, and reads as

$$\underset{1}{S}{}^{\alpha}_{\ \beta} = \frac{\nabla^2(C^{\alpha}_{\ ,\beta} + C_{\beta,}^{\ \ \alpha})}{a_0^2 C}J + \frac{8C}{3}(C^{\alpha}_{\ ,\beta} + C_{\beta,}^{\ \ \alpha})\coth(Ct) \\ -\left(B_{,\beta}^{\ \alpha} + \frac{\delta^{\alpha}_{\beta}}{3}\nabla^2 B\right)\frac{3I}{2^{4/3}a_0^2C^2}\coth(Ct) \\ +\left(A_{,\beta}^{\ \alpha} + \frac{\delta^{\alpha}_{\beta}}{3}\nabla^2 A\right)\coth(Ct). \tag{44}$$

The general solution of Eq. (43) is

$$S^{\alpha}_{\ \beta} = \underset{0}{S}{}^{\alpha}_{\ \beta} + \underset{1}{S}{}^{\alpha}_{\ \beta}, \tag{45}$$

where $\underset{0}{S}{}^{\alpha}_{\ \beta}$ is the general solution of the wave equation

$$a^2\,\underset{0}{\ddot{S}}{}^{\alpha}_{\ \beta} + 3a\dot{a}\,\underset{0}{\dot{S}}{}^{\alpha}_{\ \beta} - \nabla^2\,\underset{0}{S}{}^{\alpha}_{\ \beta} = 0 \tag{46}$$

satisfying $\underset{0}{S}{}^{\alpha\mu}_{\ \ \ ,\mu} = 0$.

Expanding the solutions of this equation in plane waves

$$\underset{0}{S}{}^{\alpha}_{\ \beta}(\boldsymbol{x}, t) = \int_0^{\infty} D^{\alpha}_{\ \beta}(\boldsymbol{k}, t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\mathrm{d}^3k, \tag{47}$$

the amplitude $D^{\alpha}_{\ \beta}(\boldsymbol{k}, t)$ of the modes satisfies

$$a^2\ddot{D}^{\alpha}_{\ \beta} + 3a\dot{a}\dot{D}^{\alpha}_{\ \beta} + (\boldsymbol{k}\cdot\boldsymbol{k})D^{\alpha}_{\ \beta} = 0 \tag{48}$$

and is transverse, $D^{\alpha\beta}k_{\beta} = 0$. Changing to the variable $x = \sinh^{2/3}(Ct)$ we get the following equation

$$x(1 + x^3)\frac{\partial^2 D}{\partial x^2} + \left(4x^3 + \frac{5}{2}\right)\frac{\partial D}{\partial x} + k^2 D = 0, \tag{49}$$

where $k^2 = \left(\frac{3}{2a_0C}\right)^2(\boldsymbol{k}\cdot\boldsymbol{k})$, with the solution

$$D = x^{-3/2}\sqrt{1 + 4k^2x + 4x^3} \\ \times[P_k\cos(\omega_k(x)) + Q_k\sin(\omega_k(x))] \tag{50}$$

and

$$\omega_k(x) = \sqrt{k^6 + \frac{27}{64}}\int_0^x \frac{\sqrt{x'}\,dx'}{(x'^3 + k^2x' + 1/4)\sqrt{1 + x'^3}}. \tag{51}$$

In summary, our solution for the linear perturbations of a $k = 0$ pressure-free FRW universe with a cosmological constant has the form

$$p = 0\,, \qquad \delta p = 0\,, \qquad h^{0\alpha} = \frac{1}{a_0}\sinh^{-2/3}(Ct)\nabla^2 C^{\alpha}, \tag{52}$$

$$h_{\alpha\beta} = \underset{0}{S}{}_{\alpha\beta} + A_{,\alpha\beta}\coth(Ct) + \eta_{\alpha\beta}B \\ -B_{,\alpha\beta}\frac{3I}{2^{4/3}a_0^2C^2}\coth(Ct) + \frac{\nabla^2(C_{\alpha,\beta} + C_{\beta,\alpha})}{a_0^2C}J \\ +\frac{8C}{3}(C_{\alpha,\beta} + C_{\beta,\alpha})\coth(Ct), \tag{53}$$

$$\delta\rho = \frac{C_M}{2a_0^3}\frac{\cosh(Ct)}{\sinh^3(Ct)}\left(\nabla^2 A - \frac{3\nabla^2 B}{2^{4/3}a_0^2C^2}I\right), \tag{54}$$

where $A = A(x^{\alpha})$, $B = B(x^{\alpha})$, $C_{\alpha} = C_{\alpha}(x^{\alpha})$ are space functions, and the time-dependent amplitudes I and J are elliptic integrals,

$$I = 2^{-2/3}\sqrt{3\Lambda}\int_0^t \frac{\sinh^{2/3}(C\tau)}{\cosh^2(C\tau)}\,\mathrm{d}\tau, \\ J = -2^{-1/3}3I(t) - 3\sinh^{-1/3}(Ct)\cosh^{-1}(Ct). \tag{55}$$

The term $\underset{0}{S}{}_{\alpha\beta}$ represents gravitational waves and is a solution of the wave Eq. (46). In obtaining this result we used the gauge freedom (5) up to transformations with harmonic generator functions b and c^{μ} satisfying

$$\nabla^2 b = 0\,, \qquad \nabla^2 c^{\mu} = 0\,, \qquad c^{\mu}_{\ ,\mu} = 0.$$

By Eq. (17) the form of the solution is not preserved by arbitrary gauge transformations.

4. The Sachs-Wolfe effect

The Sachs-Wolfe effect is the contribution to the temperature variation δT of the cosmic background radiation due to the gravitational perturbations along the path of the photon. It can be computed (Sachs & Wolfe 1967) as follows,

$$\frac{\delta T}{T} = \frac{1}{2}\int_0^{\eta_R - \eta_E}\left(\frac{\partial h_{\alpha\beta}}{\partial\eta}\mathrm{e}^{\alpha}\mathrm{e}^{\beta} - 2\frac{\partial h_{0\alpha}}{\partial\eta}\mathrm{e}^{\alpha}\right)\mathrm{d}w, \tag{56}$$

where η_R and η_E denote the time of reception and emission, respectively, and w is the affine length along the null geodesic of propagation with tangent four-vector

$$\frac{\mathrm{d}x^a}{\mathrm{d}w} = (-1, \mathrm{e}^{\alpha})\ , \tag{57}$$

where e^{α} is the three-vector of the photon direction normalized by $\mathrm{e}^{\alpha}e_{\alpha} = -1$. We consider the contribution of the relatively increasing mode. Then $h_{0\beta} = 0$ and the second term under the integral in Eq. (56) vanishes. The term $\underset{1}{S}{}_{\alpha\beta}$ has the amplitude $B_{,\alpha\beta}$. By using the relation (57) in the identical decomposition

$$y_{,a}\frac{\mathrm{d}x^a}{\mathrm{d}w}\mathrm{d}w = y_{,\alpha}\mathrm{e}^{\alpha}\mathrm{d}w - y'\mathrm{d}w \tag{58}$$

for an arbitrary function $y = y(\eta, x^{\alpha})$, we get dipole anisotropy contributions with the amplitude $B_{,\beta}\mathrm{e}^{\beta}$ and gravitational redshift terms. The time dependence of the trace part of $h_{\alpha\beta}$ is different from that in Sachs & Wolfe (1967) thus the cancellation of the integrated terms does not occur here. Taken together the contributions to the temperature variation sum up to

$$\frac{\delta T}{T} = -\frac{3}{2^{7/3}a_0^2C^2}\int_0^{\eta_R-\eta_E} B_{,\alpha\beta}e^\alpha e^\beta \Delta(t)dw \quad (59)$$
$$= -\left[\frac{3}{2^{7/3}a_0^2C^2}B_{,\alpha}e^\alpha\Delta(t) - B\Delta_{SW}(t)\right]_{\eta_E}^{\eta_R} + \left(\frac{\delta T}{T}\right)_{ISW}$$

at the respective events R of reception and E of emission, where the transfer functions of the fluctuations are defined

$$\Delta(t) = a\,(I\coth(Ct))^\bullet = a_0\sinh(Ct)^{2/3}\,(I\coth(Ct))^\bullet, \quad (60)$$

$$\Delta_{SW}(t) = \frac{3}{2^{7/3}a_0C^2}\sinh(Ct)^{2/3}\dot{\Delta}(t) \quad (61)$$

and

$$\left(\frac{\delta T}{T}\right)_{ISW} = \frac{3}{2^{7/3}C^2}\int_{\eta_E}^{\eta_R}\sinh(Ct)^{2/3}\left[B\sinh(Ct)^{2/3}\dot{\Delta}(t)\right]^\bullet dw$$
$$= -a_0C2^{1/3}\int_{\eta_E}^{\eta_R}B\Delta_{ISW}(t)dw \quad (62)$$

is the integrated Sachs-Wolfe (ISW) term with the transfer function

$$\Delta_{ISW} = \frac{2\cosh^2(Ct)+3}{3\sinh^2(Ct)}I - \frac{2^{1/3}}{\sinh^{1/3}(Ct)\cosh(Ct)}. \quad (63)$$

The terms containing $B_{,\alpha}e^\alpha$ have a dipole character (Sachs & Wolfe 1967). The relativistic redshift effect, represented by the second term in Eq. (59) is displayed in Fig. 4. The amplitude Δ_{SW} of the temperature variations decays exponentially. In the limit of a vanishing cosmological constant, the amplitude has the constant value 0.1.

5. Physical interpretation

The combined results of the *WMAP* and *SDSS* surveys (Tegmark et al. 2004a) are for the matter and dark energy densities of the Universe that $\Omega_m = 0.3 \pm 0.04$ and $\Omega_\Lambda = 0.7 \pm 0.04$, respectively. For the Hubble parameter $H_0 = h \cdot 100$ km s^{-1} Mpc^{-1} we use $h = 0.7 \pm 0.04$. From these parameters we obtain the value of the cosmological constant $\Lambda = 3H_0^2\Omega_\Lambda/c^2 = 0.12 \times 10^{-51}$ m^{-2} and the current age of the Universe $t_R = 13.47 \times 10^9$ yr. The constant C can be determined by use of (15) to be $C = 2.85 \times 10^{-18}$ s^{-1}. Taking the decoupling to occur at redshift $z = 1100$, this corresponds in the present model to $t_{dec} = 4.66 \times 10^5$ yr.

The amplitude of the density contrast at the decoupling is estimated in Kolb & Turner (1990), to be $\delta\rho/\rho \approx 10^{-3}$. A more recent estimate based on the σ_8 measurements (Tegmark et al. 2004b) confirms this order of magnitude. The current value is determined by the amplitude of the relatively growing mode

$$\left(\frac{\delta\rho}{\rho}\right)_{t_R} = \frac{(I\coth(Ct))_{t_R}}{(I\coth(Ct))_{t_{dec}}}\left(\frac{\delta\rho}{\rho}\right)_{t_{dec}} \approx 0.8. \quad (64)$$

This corresponds to an amplification factor 800. Our result is more than an order of magnitude higher then the data in Table 1 of Heath (1977). Lahav et al. (1991) find that the growth factor depends only weakly on the value of the cosmological constant, but is much more sensitive to that value for higher redshifts.

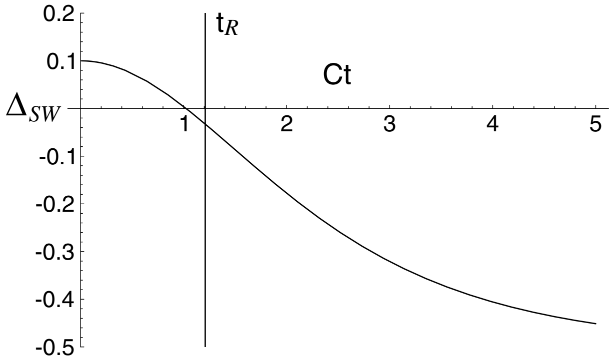

Fig. 4. The dependence of the transfer function Δ_{SW} on the distance traveled by the photon from the event of last scattering. The decoupling can be taken to occur at $Ct_{dec} = 0$ within the error bar. Reception is indicated at $Ct_R = 1.21$.

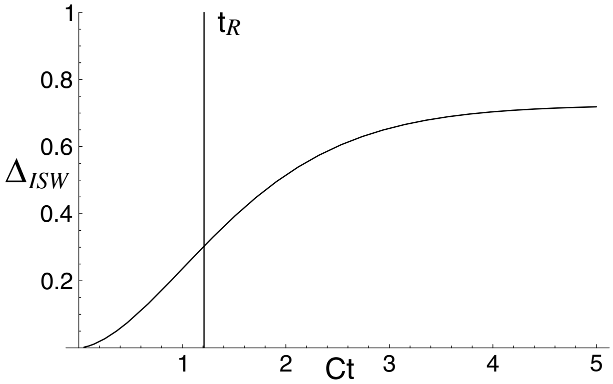

Fig. 5. The weighing function of the ISW effect.

The amplitude of the temperature fluctuations $\delta T/T$ of the CMBR due to the Sachs-Wolfe effect on the relatively growing mode vanishes at $Ct = 1.02$. At the moment of reception, the amplitude is $\Delta_{SW} = -0.033$. Thus our conclusion is that the observed angular variation of the temperature fluctuations of the CMBR are attenuated by a factor of 3 in the presence of dark energy. The temporal variation of the SW amplitude has been observed in detailed numerical studies (Hu & Sugiyama (1995)). These computations follow the variation of the multipole moments, which obscures the effect.

The weighing function of the ISW effect is displayed in Fig. 5. The dominance of the late-time contribution is apparent, in agreement with earlier predictions (Crittenden & Turok 1996). The correlation of the ISW power spectrum with the mass distribution from galaxy counts has been verified by Fosalba et al. (2003).

6. Concluding remarks

The results of our work are relevant in two major contexts: first, we find that the observed CMBR power spectrum is attenuated by a significant overall factor due to the presence of the cosmological constant. Our finding is consistent with the results of the numerical studies in Hu & Sugiyama (1995). This phenomenon may help understanding structure formation by taking into account a higher level of the initial matter fluctuations. The second observation of interest that we make is the natural complexification of the perturbed fields in the normal picture of Legendre. This complex description may prove useful in a treatment of quantum fluctuations, a phenomenon that occurs also in the Fourier expansion of the wave solutions.

7. Appendix: Legendre normal forms

The integral (29) can be brought to the Legendre normal form (Grobner & Hofreiter 1957)

$$I = \frac{1}{2^{1/6}}\left(3+3^{1/2}i\right)^{1/2}\left(E - \frac{3+3^{1/2}i}{6}F\right) - \frac{2^{1/3}(1-x)\,x^{1/2}}{(x^3+1)^{1/2}} \tag{65}$$

using the mirror symmetries of the incomplete elliptic integrals

$$E = E\left(\left[\frac{(3+3^{1/2}i)x}{2(x+1)}\right]^{1/2}\middle|\frac{i-3^{1/2}}{2}\right) \tag{66}$$

and

$$F = F\left(\left[\frac{(3+3^{1/2}i)x}{2(x+1)}\right]^{1/2}\middle|\frac{i-3^{1/2}}{2}\right)\cdot \tag{67}$$

To get the Legendre form of

$$J = \int_0^w \frac{\mathrm{d}w'}{w'^{4/3}\,(1+w'^2)^{1/2}} = \frac{3}{2}\int_0^x \frac{\mathrm{d}x'}{x'^{3/2}\,(1+x'^3)^{1/2}} \tag{68}$$

we use the variable $w = x^{3/2}$ in the integral. The normal form is (Grobner & Hofreiter 1957)

$$J = -\frac{3}{2}2^{1/2}(3+3^{1/2}i)^{1/2}\left(E - \frac{3+3^{1/2}i}{6}F\right) - 3\frac{(x^2-x+1)^{1/2}}{(x+1)^{1/2}x^{1/2}}\cdot \tag{69}$$

The following relation holds between the incomplete elliptic integrals:

$$J + \frac{3}{2^{1/3}}I = -\frac{3}{x^{1/2}(1+x^3)^{1/2}}\cdot \tag{70}$$

Acknowledgements. We thank professor Jürgen Ehlers for suggesting improvements to the manuscript. This work was supported by OTKA Nos. T031724 and TS044665 grants.

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